Wakefields and Impedances

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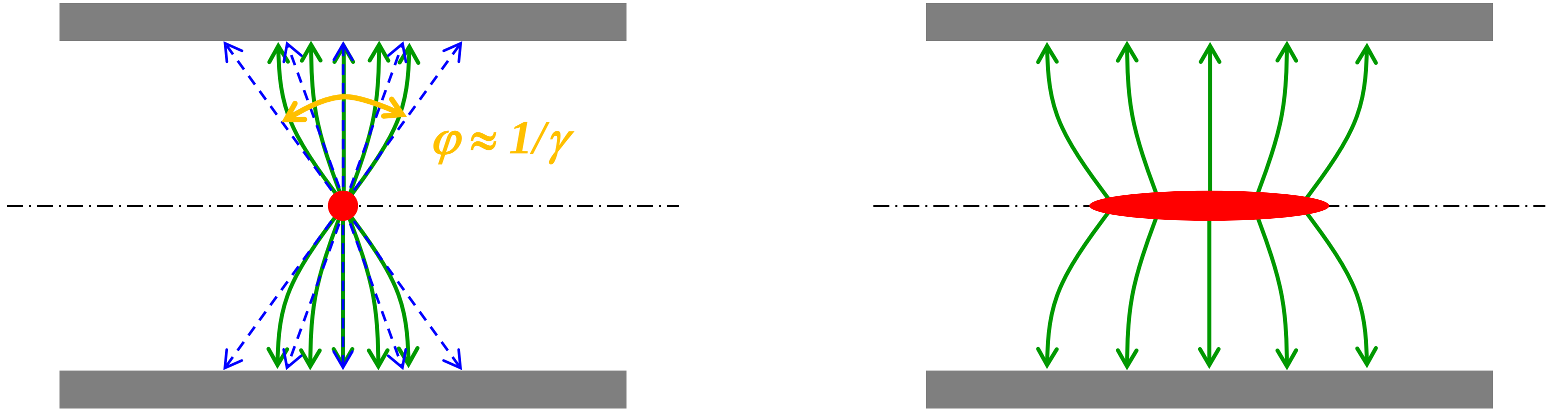
Introduction

Space Charge Force

infinite perfect conducting beam pipe



collective motion ≈ uniform into z direction

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}(\mathbf{r}-\mathbf{e}_z v_c t)\\ \mathbf{B}(\mathbf{r},t) &= c^{-2}\mathbf{e}_z v_c \times \mathbf{E}(\mathbf{r}-\mathbf{e}_z v_c t)\end{aligned}\right\} \quad \mathbf{F} = q_0\left(\mathbf{E} + \boxed{\mathbf{v}} \times \mathbf{B}\right)$$

test particle

$$\boldsymbol{\beta} = \boxed{\begin{pmatrix} x' \\ y' \end{pmatrix}}\!\!\begin{matrix} \\ \\ 1 \end{matrix}\left(\beta_c + \frac{\boxed{\Delta\gamma}}{\beta_c \gamma_c^3}\right) + O^2 \rightarrow \qquad F_\perp \approx \frac{q_0}{\boxed{\gamma_c^2}}\left(1 + \frac{\Delta\gamma}{\gamma_c}\right) E_\perp$$

$$F_z \approx q_0 E_z + q_0 \beta_c^2 \left(x' E_x + y' E_y\right)$$

in many cases:

$$x', y' \propto \gamma_c^{-1/2}$$

Wake Function

approximations: $v \to c$
constant offset
constant distance

source particle q_1 $\mathbf{r}_1(t) = x_1\mathbf{e}_x + y_1\mathbf{e}_y + ct\,\mathbf{e}_z$

causes electromagnetic field $\mathbf{E}^{(1)}(\mathbf{r},t), \mathbf{B}^{(1)}(\mathbf{r},t),$

this field particle is observed by a test particles q_2 $\mathbf{r}_2(t) = x_2\mathbf{e}_x + y_2\mathbf{e}_y + (ct - s)\mathbf{e}_z$

$$\mathbf{E}^{(1)}(\mathbf{r},t), \mathbf{B}^{(1)}(\mathbf{r},t),$$

$$\mathbf{F}_2 = q_2\left(\mathbf{E}^{(1)}(\mathbf{r}_2,t) + \mathbf{v}_2 \times \mathbf{B}^{(1)}(\mathbf{r}_2,t)\right)$$

change of momentum $\Delta\mathbf{p} = \int \mathbf{F}dt$

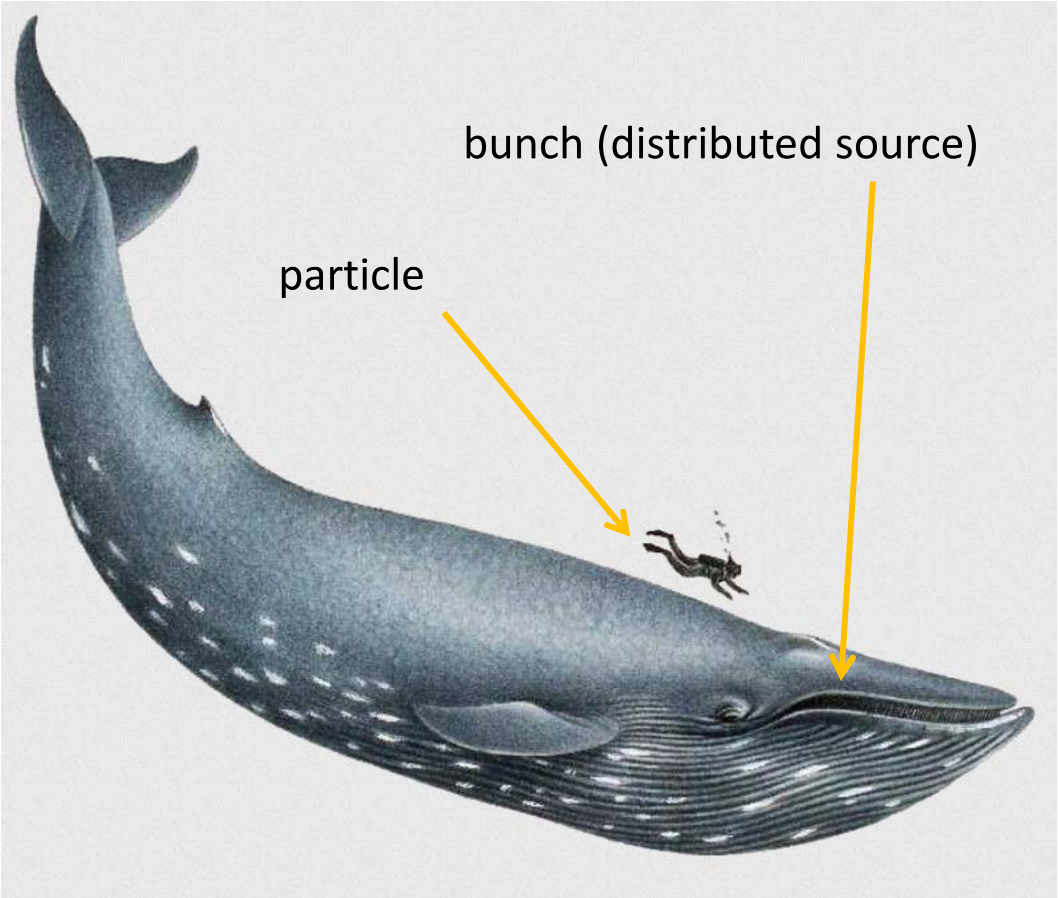
wake function

$$\mathbf{w}(x_1,y_1,x_2,y_2,s) = \frac{\Delta\mathbf{p}}{q_1 q_2} = \frac{1}{q_1}\int_{-\infty}^{\infty} dz \times \left[\mathbf{E}^{(1)}(x_2,y_2,z,t) + c\mathbf{e}_z \times \mathbf{B}^{(1)}(x_2,y_2,z,t)\right]_{t=(s+z)/c}$$

it is a Green's function (point to point)

Wake Potential

rigid beam approximation



source $\rho(\mathbf{r},t) = \rho_0(x - x_1, y - y_1, z - ct)$ shape $\rho_0(x,y,z)$ and offset x_1, y_1

wake potential $$\mathbf{W}(x_1,y_1,x_2,y_2,s) = \frac{1}{q_1}\int_{-\infty}^{\infty} dz \times \left[\mathbf{E}^{(1)}(x_2,y_2,z,t) + c\mathbf{e}_z \times \mathbf{B}^{(1)}(x_2,y_2,z,t)\right]_{t=(s+z)/c}$$

direct calculation in time domain
computer codes as PBCI, Echo, ABCI, … MAFIA

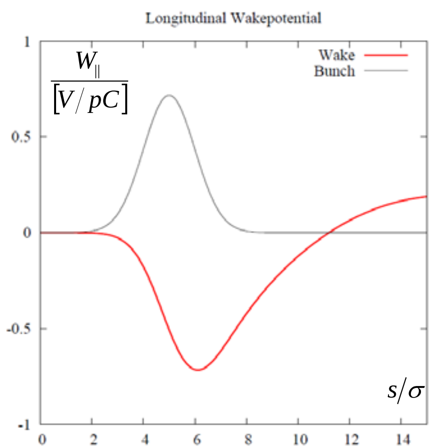
Wake Potential

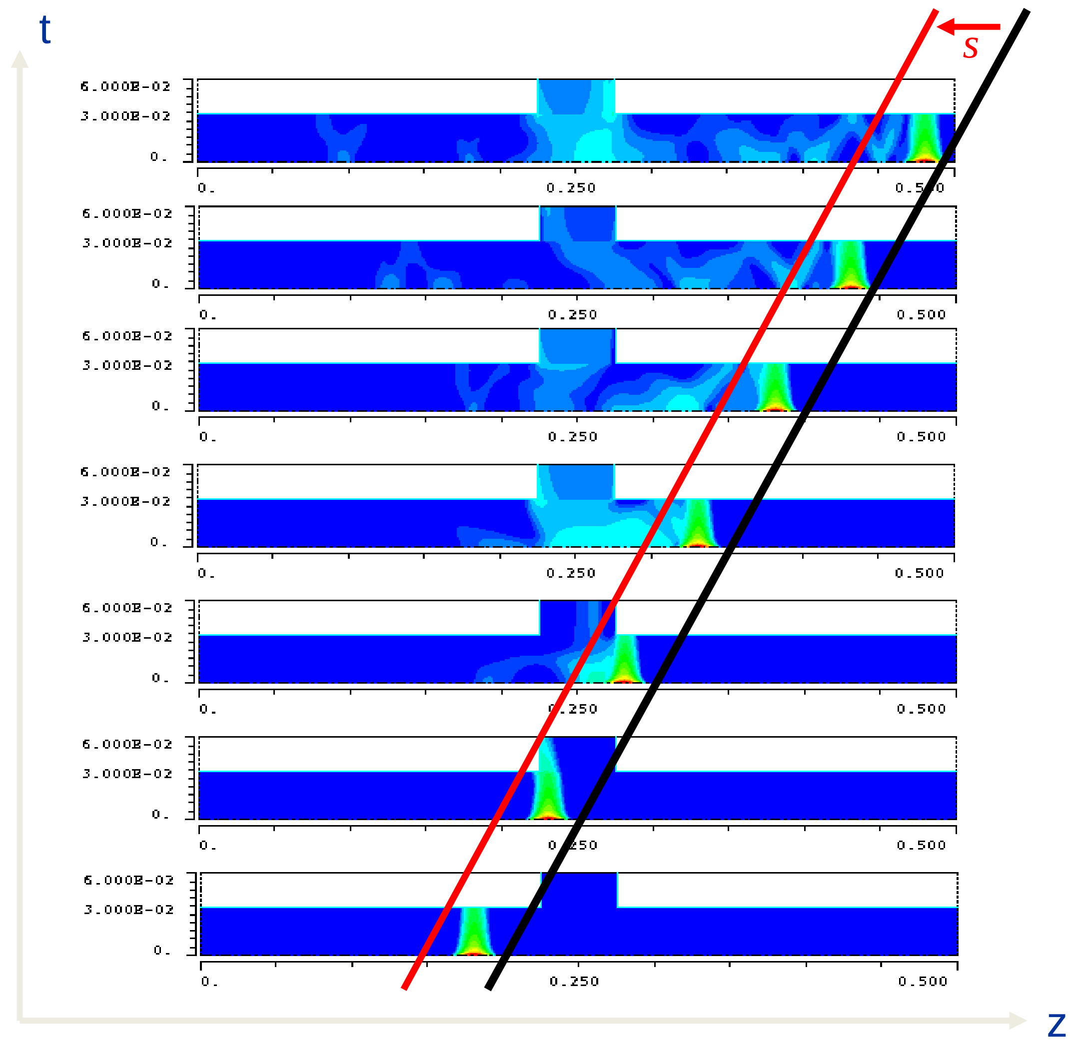
example: pill-box cavity in rz coordinates

longitudinal wake

$$W_{\parallel}(\cdots,s)=\frac{1}{q_1}\int_{-\infty}^{\infty}E^{(1)}(\cdots,z,t(s,z))\,dz$$

short range: causal, energy loss





long range: oscillations, de- & recoherence , decay

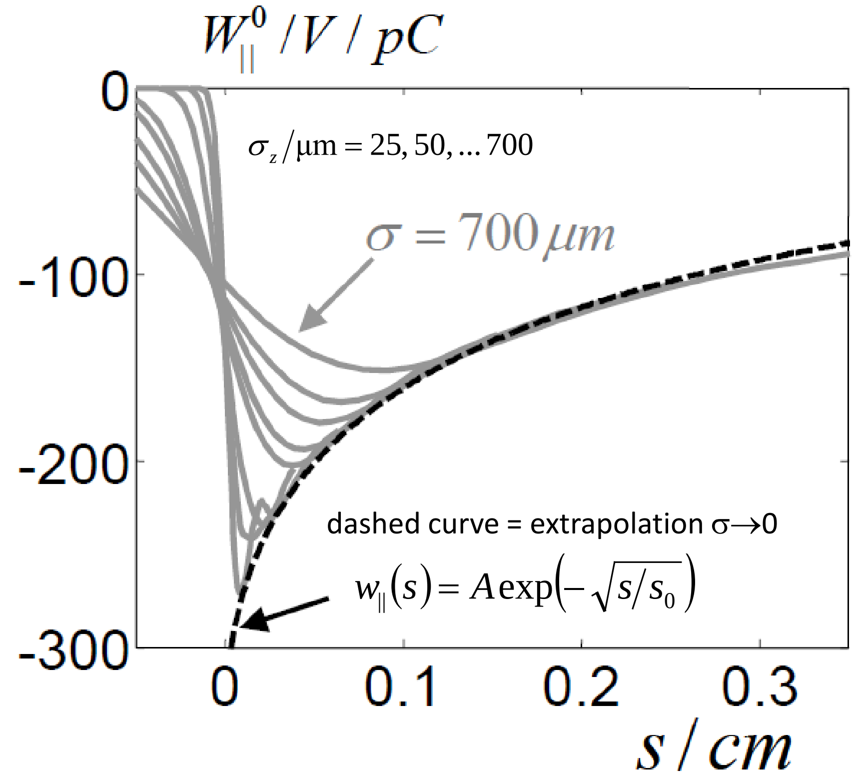
de- and recoherence is the interference of multiple cavity modes
decay is caused by damping (wall losses & absorbers)

Potential → Function

by extrapolation

example: Tesla cavity, short range

(wake per cavity in infinite string of cavities)



Function → Potential

$$\rho(\mathbf{r},t) = \lambda(z-ct)\ \eta(x,y)$$

longitudinal shape f.i. $\lambda(z) = \frac{q_1}{\sqrt{2\pi}\sigma_{\|}} \exp\left(-\frac{1}{2}\left(\frac{z}{\sigma_{\|}}\right)^2\right)$

transverse shape $\eta(x,y)$

$$\mathbf{W}(x_1,y_1,x_2,y_2,s) = \int dx_1 dy_1 dz_1 \times \mathbf{w}(x,y,x_2,y_2,s+z_1)\frac{\lambda(z_1)}{q_1}\eta(x_1-x,y_1-y)$$

Total Loss Parameter

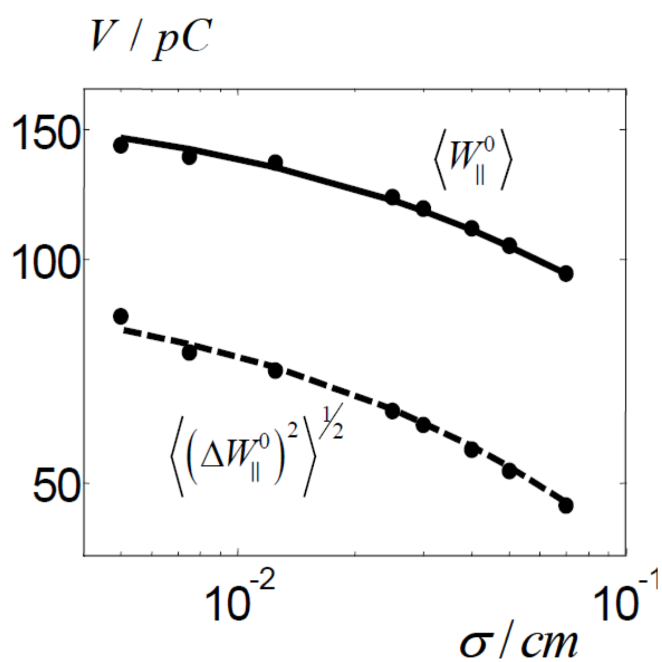
for a bunch with small transverse dimensions

bunch shape $\lambda(z-ct)$

longitudinal wake $W_{\|}(s) \approx \mathbf{e}_{\|} \cdot \mathbf{W}(\cdots, s)$

$$k_{tot} = \frac{\Delta E}{q^2} = -\int_{-\infty}^{\infty} W(z)\lambda(z)dz$$

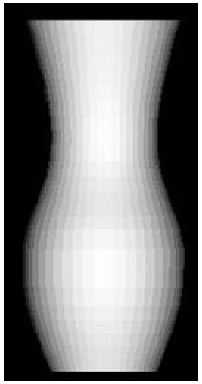
total lost energy ΔE > 0, depends on bunch shape

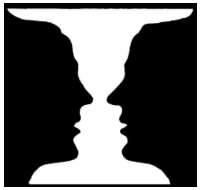


example: Tesla cavity
(infinite string of cavities)

$$\left.\begin{array}{l} q = 1\,\mathrm{nC} \\ \sigma \to 0 \end{array}\right\} \begin{array}{l} k_{tot} \to 150\,\mathrm{V/pC} \\ \Delta E \approx 0.15\,\mathrm{mJ} \\ V = \Delta E / q \to 150\,\mathrm{kV} \end{array}$$

Symmetry of Revolution





polar coordinates $x_1 = r_1 \cos\varphi_1 \;,\; y_1 = r_1 \sin\varphi_1 \;\cdots$

azimuthal expansion

$$w_{\parallel}(\cdots,s) = \sum_{m=0}^{\infty} r_1^{\,m} r_2^{\,m} w_{\parallel}^{(m)}(s)\cos(m\varphi_2 - m\varphi_1)$$

or wake potential

$$W_{\parallel}(\cdots,s) = \sum_{m=0}^{\infty} r_1^{\,m} r_2^{\,m} W_{\parallel}^{(m)}(s)\cos(m\varphi_2 - m\varphi_1)$$

Panofsky-Wenzel theorem relates transverse and longitudinal wake

$$w_x(\cdots,s) = -\int_{-\infty}^{s} \frac{d}{dx_2} w_{\parallel}(\cdots,\tilde{s})\,d\tilde{s}$$

$$w_x(\cdots,s) = \sum_{m=0}^{\infty} m r_1^{\,m} r_2^{\,m-1} w_{\perp}^{(m)}(s)\cos(m\varphi_2 - (m-1)\varphi_1)$$

$$w_y(\cdots,s) = \sum_{m=0}^{\infty} m r_1^{\,m} r_2^{\,m-1} w_{\perp}^{(m)}(s)\sin(m\varphi_2 - (m-1)\varphi_1)$$

with $w_{\perp}^{(m)}(s) = -\int_{-\infty}^{s} w_{\parallel}^{(m)}(\tilde{s})\,d\tilde{s}$

Monopole and Dipole Wakes

general 3D: wake is a 5D vector function

symmetry of revolution: two 1D scalar functions describe nearly everything

$$w_{\parallel}(x_1,y_1,x_2,y_2,s) \approx w_{\parallel}^{(0)}(s) + r_1 r_2 w_{\parallel}^{(1)}(s)\cos(\varphi_2-\varphi_1) = w_{\parallel}^{(0)}(s) + (x_1x_2+y_1y_2)w_{\parallel}^{(1)}(s)$$

$$w_x(x_1,y_1,x_2,y_2,s) \approx r_1 w_{\perp}^{(1)}(s)\cos(\varphi_2) = x_1 w_{\perp}^{(1)}(s)$$

$$w_y(x_1,y_1,x_2,y_2,s) \approx r_1 w_{\perp}^{(1)}(s)\sin(\varphi_2) = y_1 w_{\perp}^{(1)}(s)$$

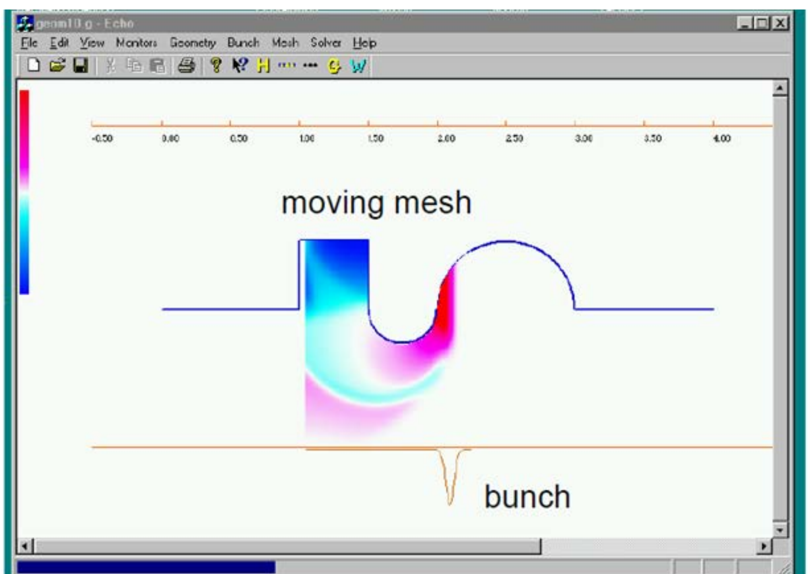
with

$$w_{\perp}^{(1)}(s) = -\int_{-\infty}^{s} w_{\parallel}^{(1)}(\tilde{s})\,d\tilde{s}$$

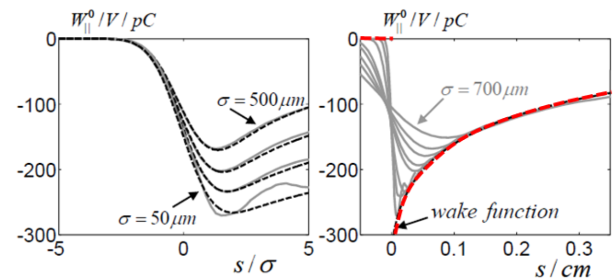
a simple job for rz wake codes, as

Echo 2D, Igor Zagorodnov http://www.desy.de/~zagor/WakefieldCode_ECHOz/

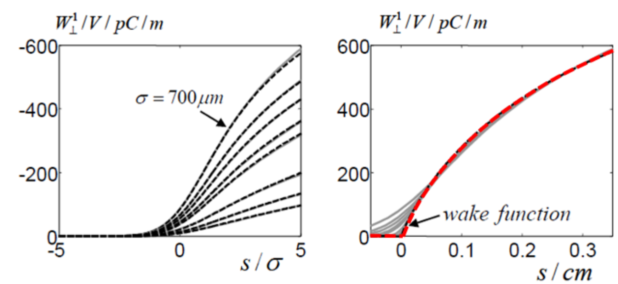
(for any azimuthal order)



longitudinal wake (monopole)



transverse wake (dipole)



General Geometry

programs as Mafia, MWS, GdfidL, Tau3P, Echo3D, PBCI, CST

$$w_{\|}(x_1,y_1,x_2,y_2,s)=\begin{bmatrix}1\\x_1\\y_1\\x_2\\y_2\end{bmatrix}^t\begin{bmatrix}w_{00}(s) & w_{01}(s) & w_{02}(s) & w_{03}(s) & w_{04}(s)\\0 & w_{11}(s) & w_{12}(s) & w_{13}(s) & w_{14}(s)\\0 & w_{12}(s) & w_{22}(s) & w_{23}(s) & w_{24}(s)\\0 & w_{13}(s) & w_{23}(s) & w_{33}(s) & w_{34}(s)\\0 & w_{14}(s) & w_{24}(s) & w_{34}(s) & w_{44}(s)\end{bmatrix}\begin{bmatrix}1\\x_1\\y_1\\x_2\\y_2\end{bmatrix}+\cdots$$

Panofsky-Wenzel theorem $\tilde{w}_{ij}(s)=-\int_{-\infty}^{s}w_{ij}(\tilde{s})d\tilde{s}$

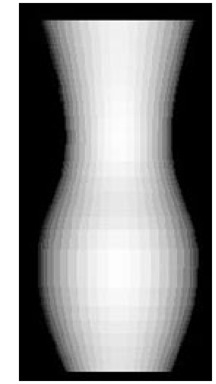
$$w_x(x_1,y_1,x_2,y_2,s)=\tilde{w}_{03}(s)+2\tilde{w}_{13}(s)x_1+2\tilde{w}_{23}(s)y_1+2\tilde{w}_{33}(s)x_2+2\tilde{w}_{34}(s)y_2+\cdots$$
$$w_y(x_1,y_1,x_2,y_2,s)=\tilde{w}_{04}(s)+2\tilde{w}_{14}(s)x_1+2\tilde{w}_{24}(s)y_1+2\tilde{w}_{34}(s)x_2+2\tilde{w}_{44}(s)y_2+\cdots$$

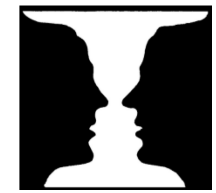
more properties $(\partial_{x_1}^2+\partial_{y_1}^2)w_{\|}=0 \quad \rightarrow w_{11}=w_{22}$

$(\partial_{x_2}^2+\partial_{y_2}^2)w_{\|}=0 \quad \rightarrow w_{33}=w_{44}$

13 coefficient functions!

Symmetry of Revolution





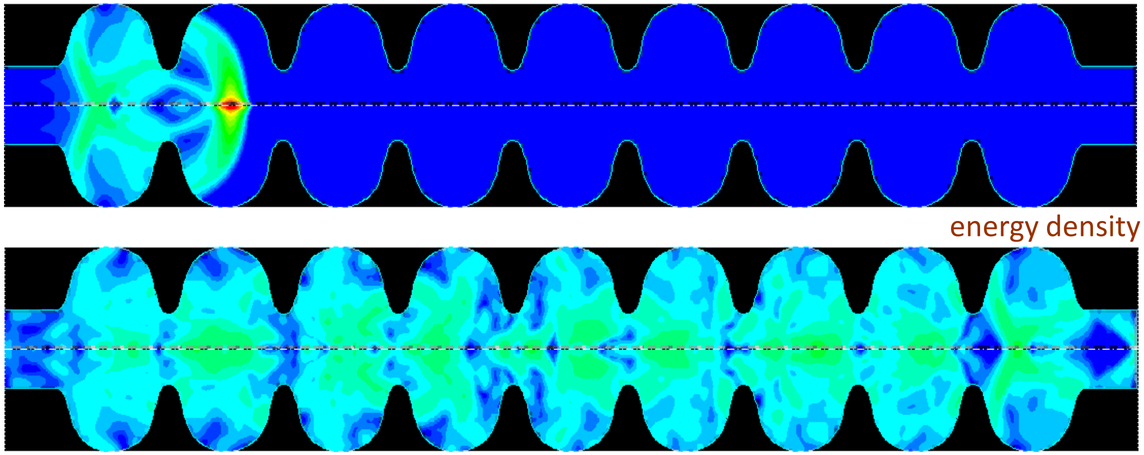
$$w_{\parallel}(x_1, y_1, x_2, y_2, s) \approx \begin{bmatrix} 1 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}^t \begin{bmatrix} w_{\parallel}^{(0)}(s) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} w_{\parallel}^{(1)}(s) & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} w_{\parallel}^{(1)}(s) \\ 0 & \frac{1}{2} w_{\parallel}^{(1)}(s) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} w_{\parallel}^{(1)}(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}$$

$$w_x(x_1, y_1, x_2, y_2, s) \approx w_{\perp}^{(1)}(s) x_1$$
$$w_y(x_1, y_1, x_2, y_2, s) = w_{\perp}^{(1)}(s) y_1$$

2 coefficient functions

Long Range Wake

example: bunch on axis of Tesla cavity excites only monopole modes



for short bunches:

total loss-parameter

$$k_{tot} \to 150\ \mathrm{V/pC}$$

accelerating mode

$$k_9 \approx 2.2\ \mathrm{V/pC}$$

$$Q_0 \sim 10^{10}$$

$$Q_L \sim 10^{6}$$

eigenmodes are excited and ring forever (if there is no damping)

International Workshop
Ultrafast Beams and Applications

$$w_{\|}^{(0)}(s>0) = \boxed{-\sum_{\nu} 2k_\nu \cos\left(\frac{\omega_\nu}{c}s\right)} + \boxed{w_{\|,r}^{(0)}(s)}$$

resonant part	residual, transient part
resonances, eigenmodes	pulses
long range effects	short range effects
impedance	wake
frequency domain	time domain

Impedance

just a Fourier transformation

$$Z_\parallel(\cdots,\omega) = -\frac{1}{c}\int_{-\infty}^{\infty} w_\parallel(\cdots,s)\exp(-i\omega s/c)ds$$

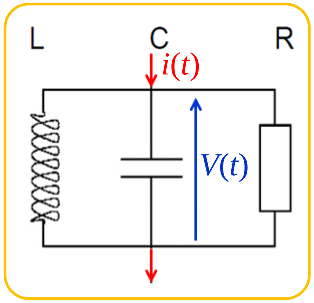
monopole modes

$$w_\parallel^{(0)}(s>0) = -\sum_\nu 2k_\nu \cos\left(\frac{\omega_\nu}{c}s\right) + w_{\parallel,r}^{(0)}(s)$$

$$Z_\parallel^{(0)}(\omega) = \sum_\nu \frac{2k_\nu i\omega}{\omega_\nu^2 - \omega^2 + i\omega\omega_\nu/Q_\nu} + Z_{\parallel,r}^{(0)}(\omega)$$

with $Q_\nu \to \infty$

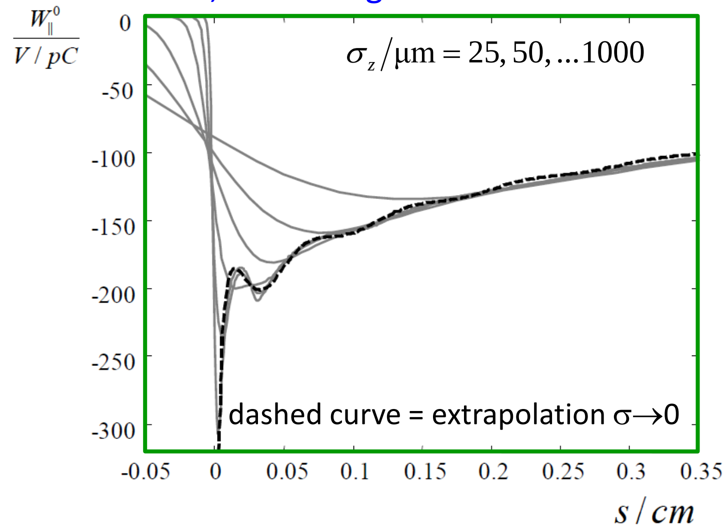
(ideal cavity)

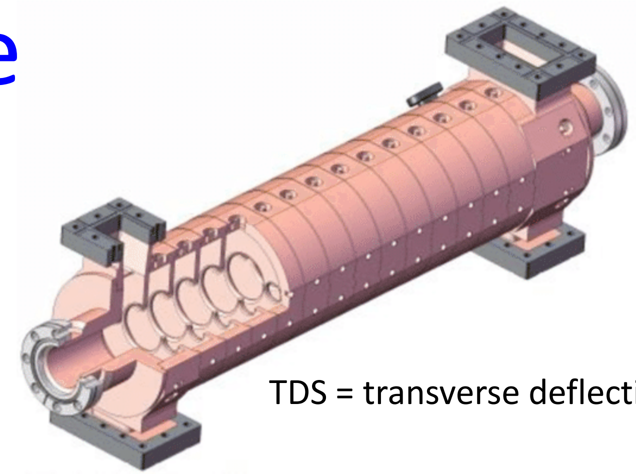


$$R_\nu = \frac{2k_\nu Q_\nu}{\omega_\nu}$$ shunt impedance

Wake ↔ Impedance

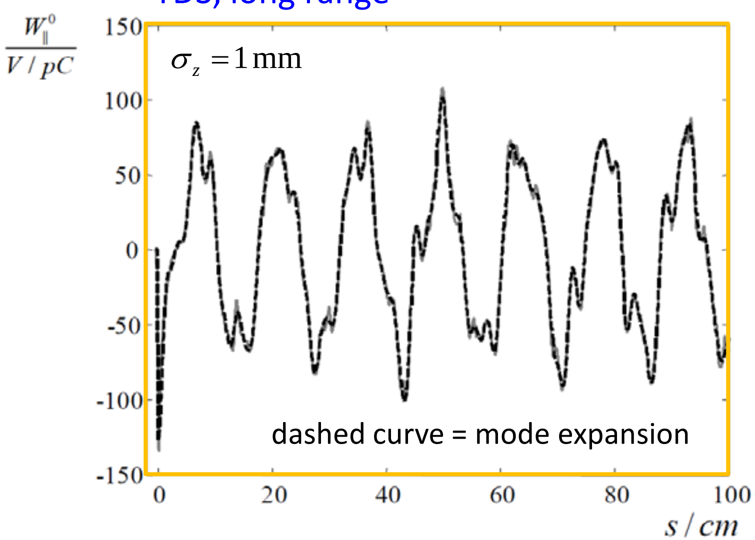
TDS, short range



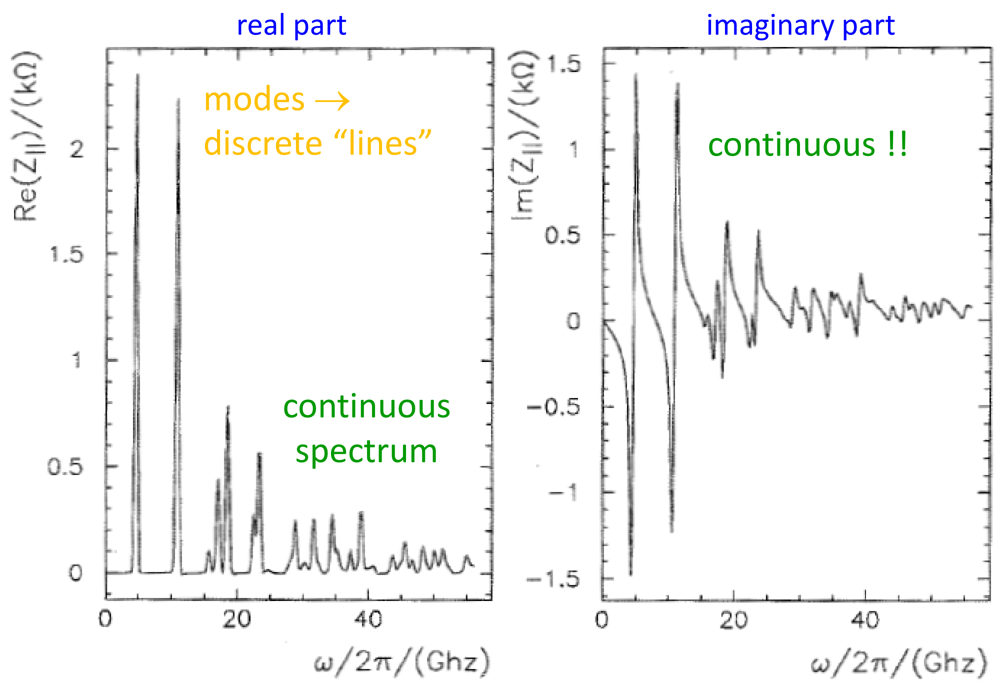


TDS = transverse deflecting structure

TDS, long range



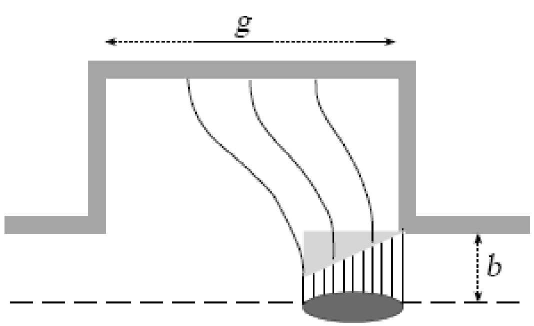
impedance of a cavity



Some Impedances

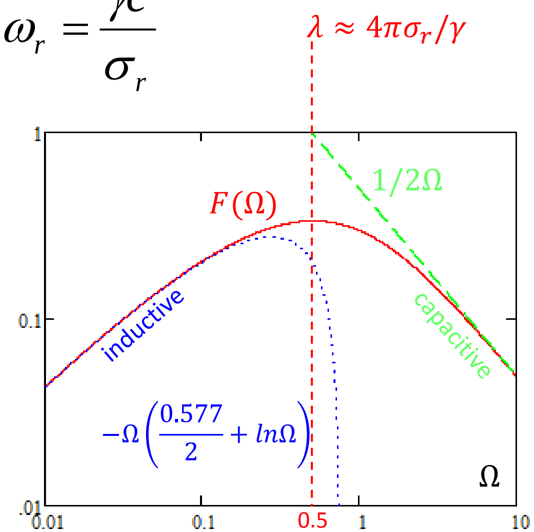
effective (averaged) impedance $Z_{\text{eff}}(\omega)=\int \eta(x_1,y_1)\eta(x_2,y_2)Z_{\|}(x_1,y_1,x_2,y_2,\omega)dx_1dy_1dx_2dy_2$

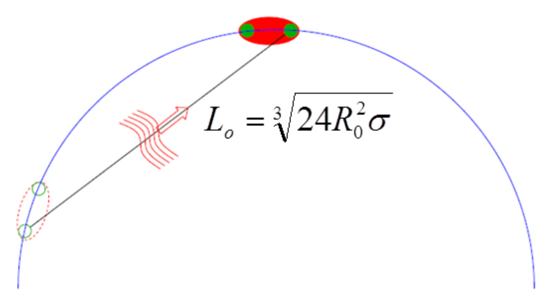
uniform motion		
response of geometry or matter	space charge	radiation
cavity gap	free space	csr
$Z(\omega)=(1+i)\frac{Z_0}{2\pi b}\sqrt{\frac{gc}{\omega\pi}}+\cdots$	$Z'_{sc}(\omega)=-\frac{iZ_0}{2\pi\sigma_r\gamma}F\left(\frac{\omega}{\omega_r}\right)$	$Z'_{CSR}(\omega)\approx 0.15\,Z_0\sqrt[3]{\frac{\omega}{icR_0^2}}$
	$\omega_r=\frac{\gamma c}{\sigma_r}$	for $\omega<<\frac{c\gamma^3}{R_0}$



resistive

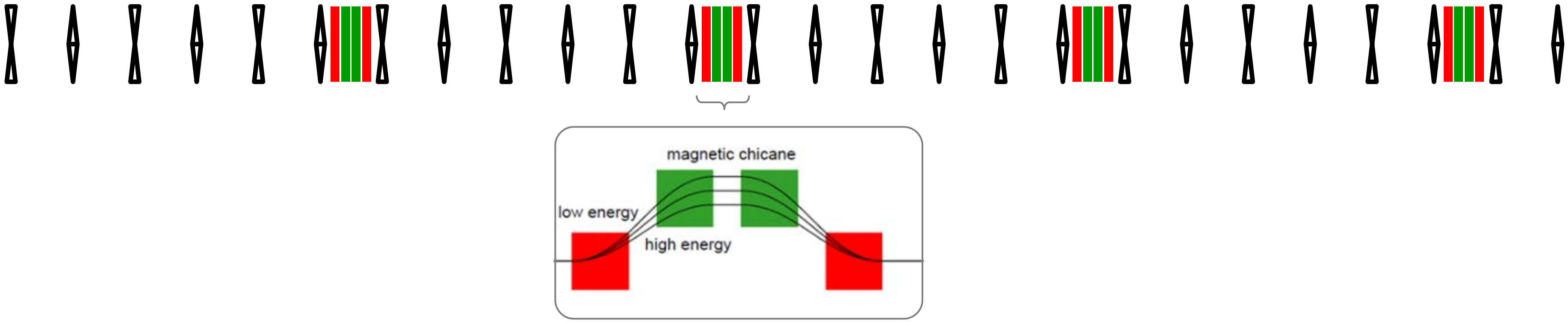
…



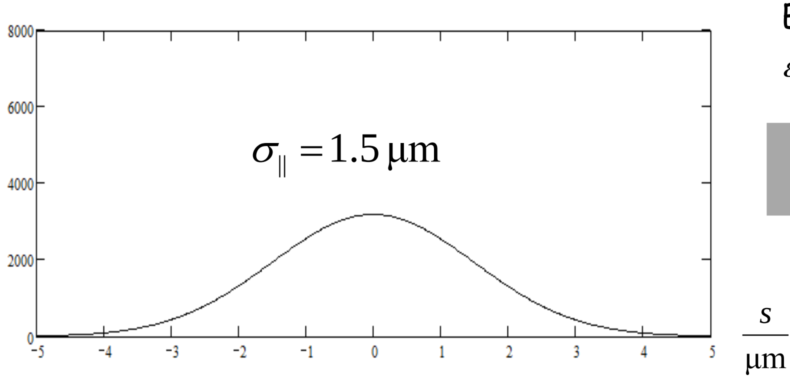


Micro-Bunching

a beam guiding structure (FODO) + some longitudinal dispersion (chicanes)



bunch current / A



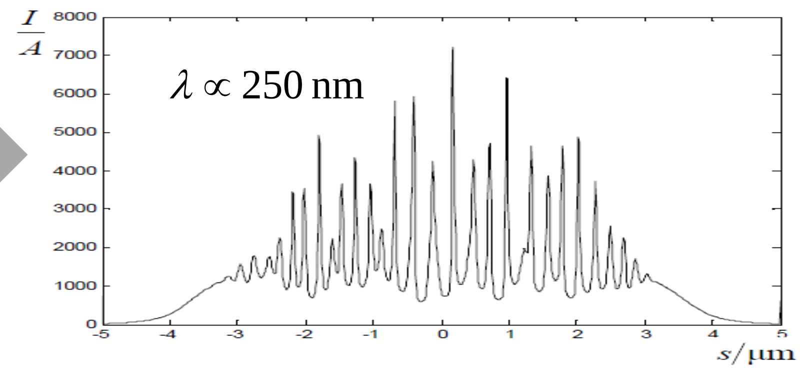
bunch coordinate

$E_{av} = 300\ \text{MeV}$

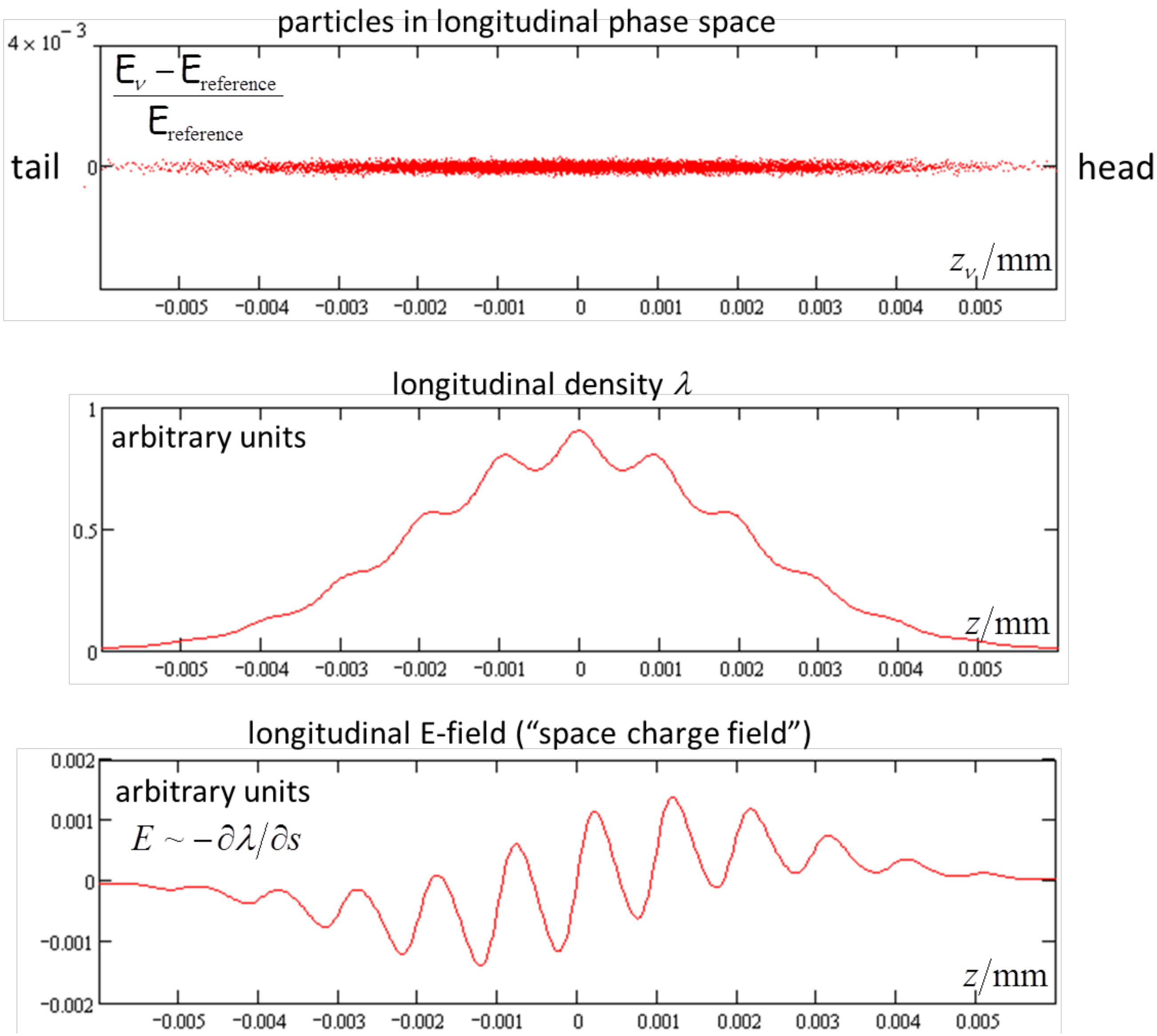
$E_{rms} = 0.6\ \text{MeV}$

$\varepsilon_n = 0.2\ \mu\text{m}$

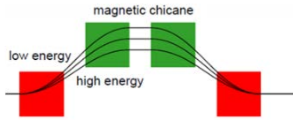
bunch current / A

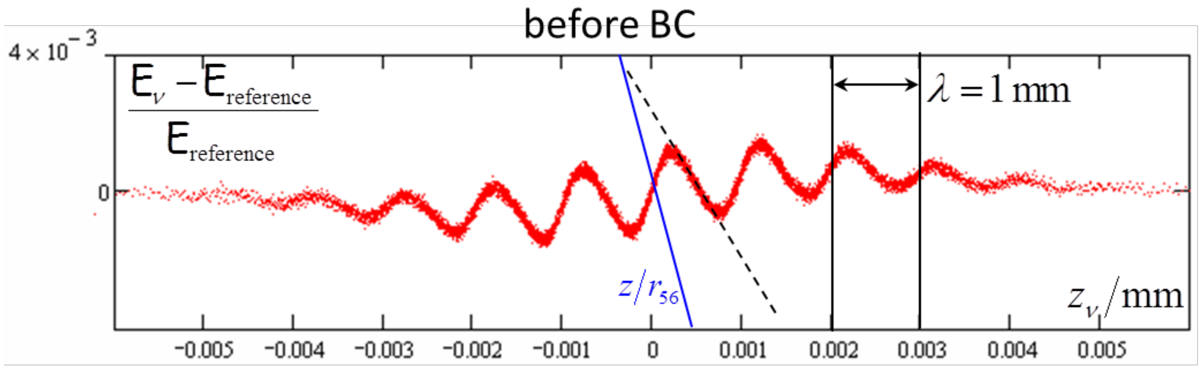


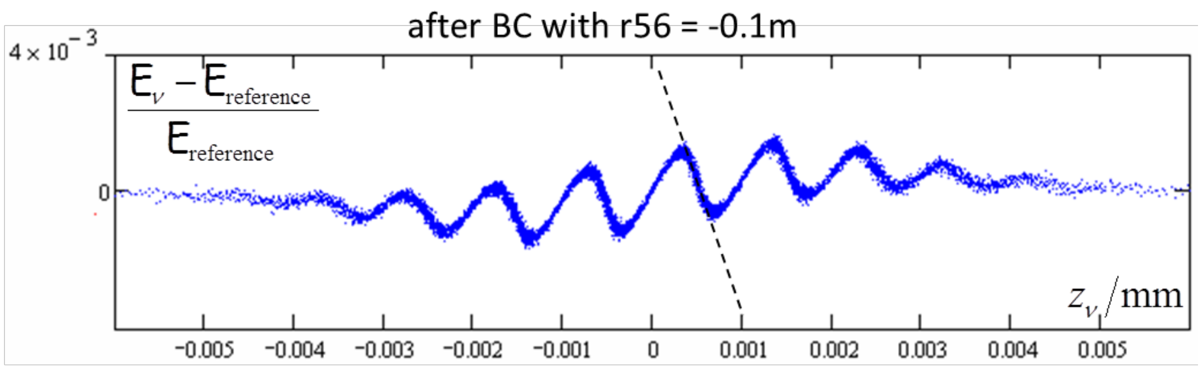
SC impedance for “weak” density modulation

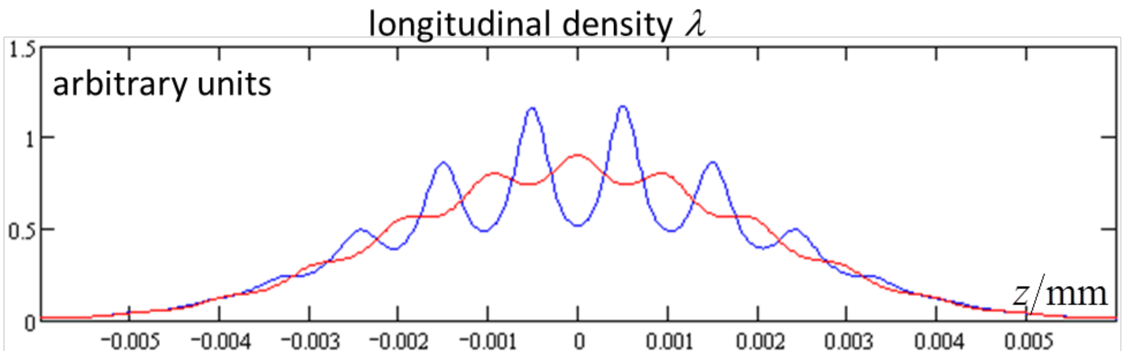


some longitudinal dispersion $s_2 = s_1 + r_{56} \frac{\Delta\gamma}{\gamma_0}$ f.i. by a 4 magnet chicane

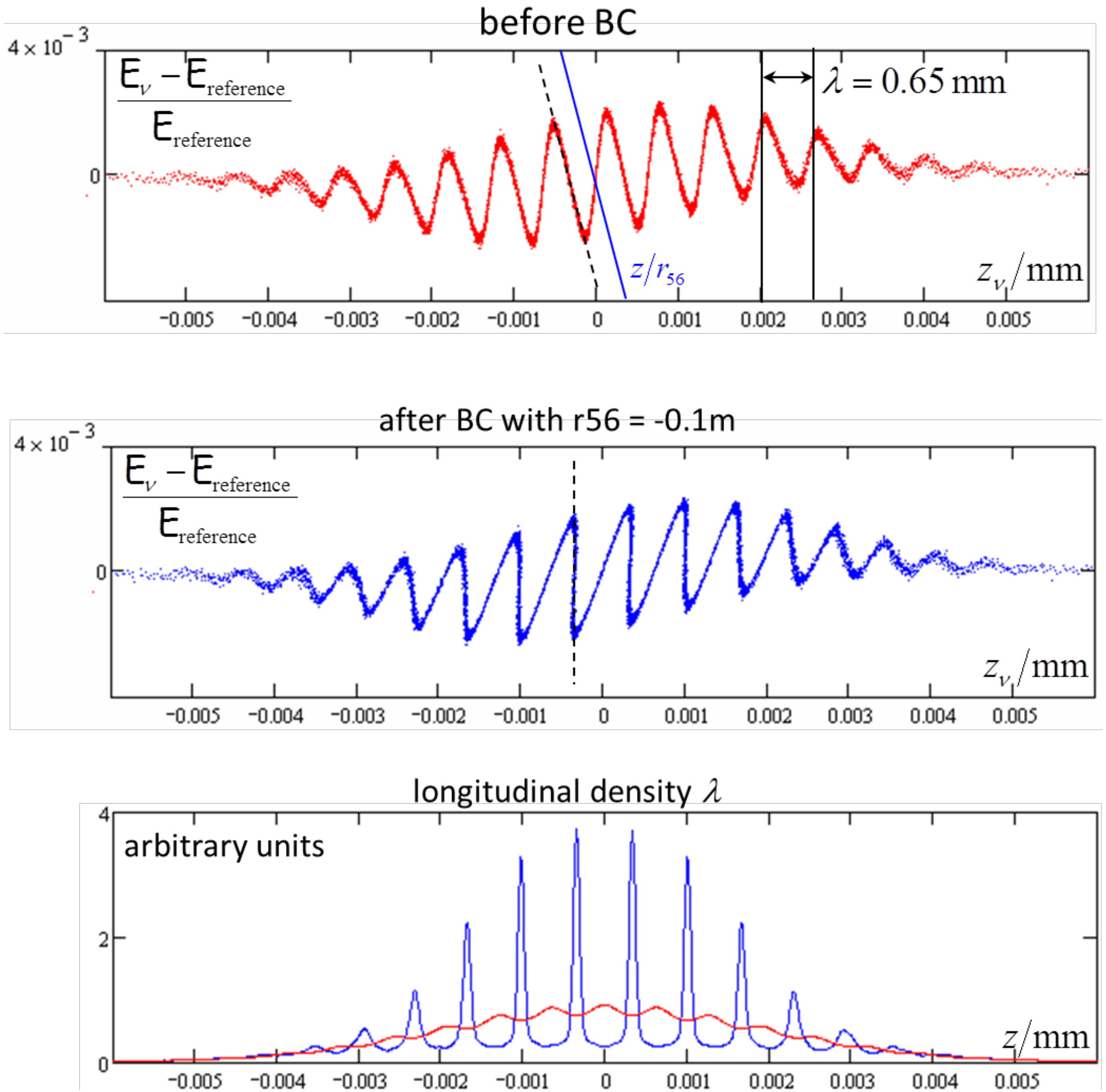






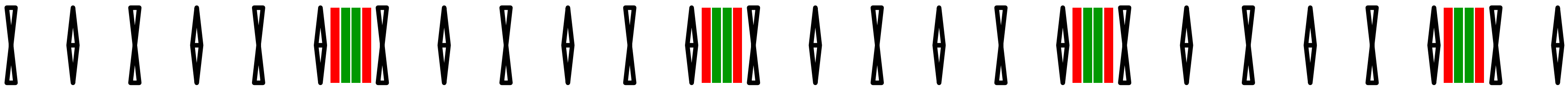


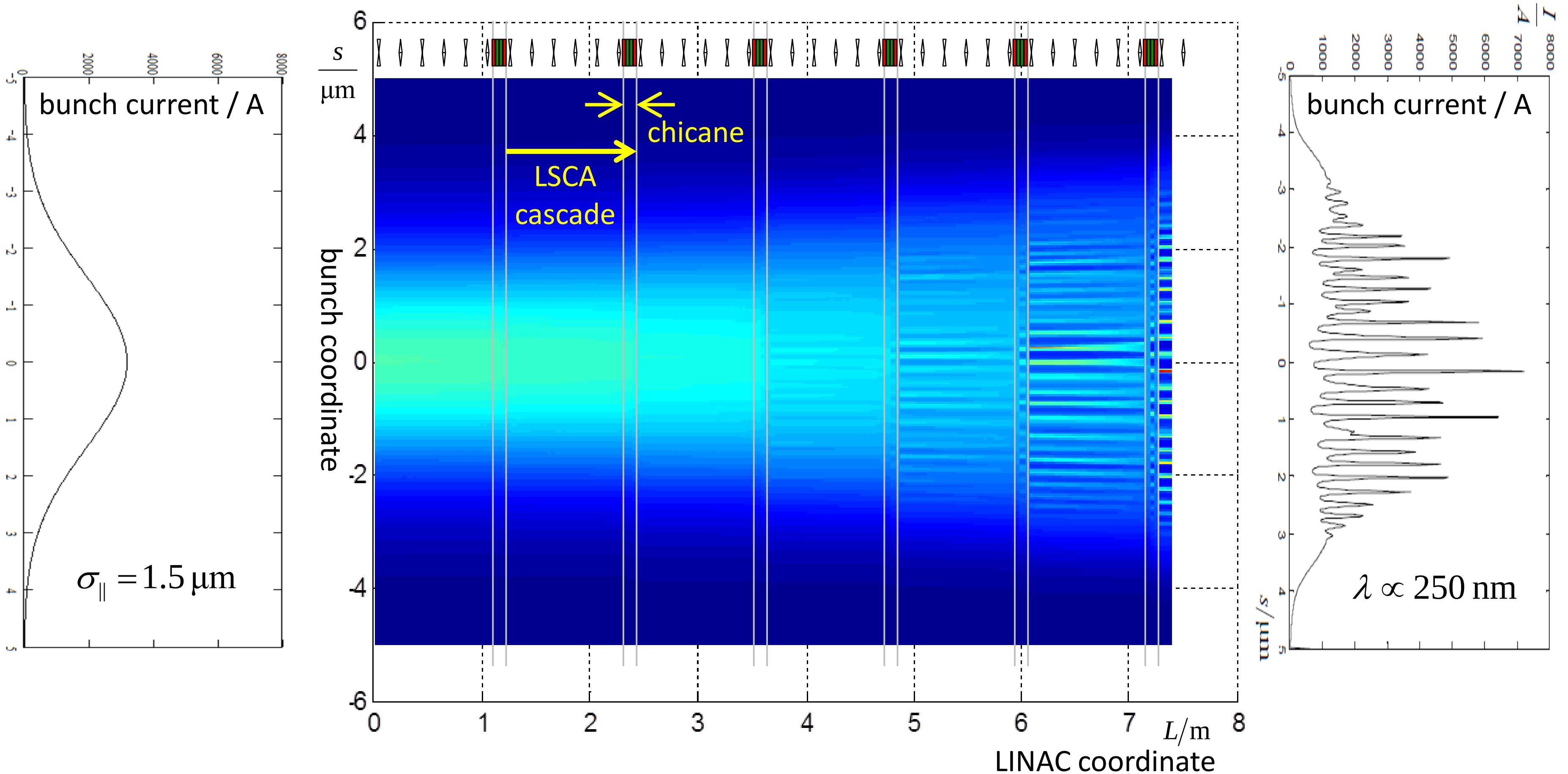
full compression $\frac{\partial}{\partial s_1}\left(s_1 + r_{56}\frac{\Delta\gamma(s_1)}{\gamma_0}\right)=0$



Multi Stage SC Amplifier

chicanes in FODO structure



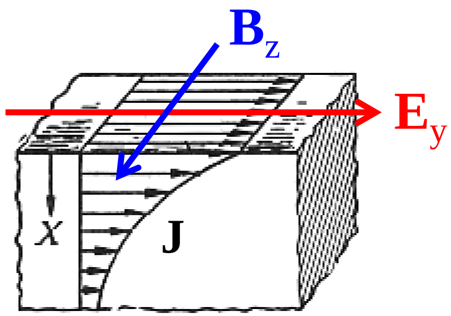


Surface Impedance

local relation of tangential fields in surface

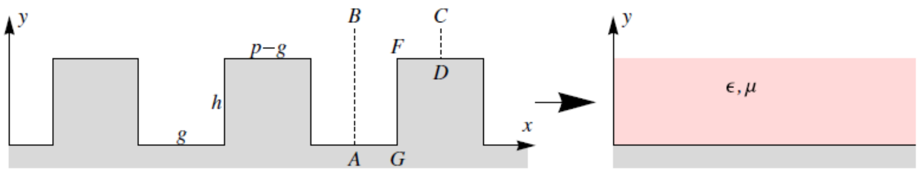
$$Z_s = \frac{E_y}{H_z} = -\frac{E_z}{H_y}$$ for a surface with $\mathbf{n} = \mathbf{e}_x$

metal surfaces $\rightarrow$ skin effect (nearly independent on curvature of surface)

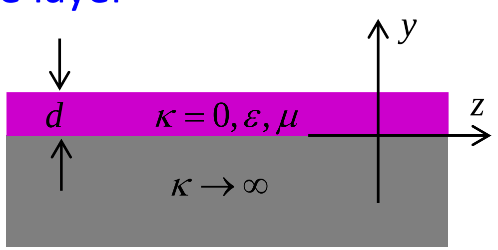


$$Z_s = \sqrt{\frac{j\omega\mu}{\kappa(\omega) + j\omega\varepsilon}}$$

artificial or random micro structure



thin surface layer



$$Z_s = j\omega d\left(\mu - \frac{k_p^2}{\omega^2 \varepsilon}\right)$$

Round Beam Pipe

monopole & dipole impedance (per length)

$$\mathbf{Z}^{(m)}(x_1,y_1,x_2,y_2,\omega)=\frac{Z_s}{2\pi R}\frac{1}{1+jk\frac{R}{2}\frac{Z_s}{Z_0}}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

(under many conditions) $\approx$

$$\mathbf{Z}^{(d)}(x_1,y_1,x_2,y_2,\omega)=\frac{Z_s}{2\pi R}\frac{1}{1+j\left(\frac{kR}{2}+\frac{1}{kR}\right)\frac{Z_s}{Z_0}+\left(\frac{Z_s}{Z_0}\right)^2}\frac{2}{R^2}\begin{pmatrix}jx_1/k\\jy_1/k\\xx_2+yy_2\end{pmatrix}$$

monopole & dipole wake (per length)

$$\mathbf{w}^{(m)}(x_1,y_1,x_2,y_2,s)=\begin{pmatrix}0\\0\\w_r(s)\end{pmatrix}$$

$$\mathbf{w}^{(d)}(x_1,y_1,x_2,y_2,s)=\frac{2}{R^2}\begin{pmatrix}x_1w_\perp^{(1)}(s)\\y_1w_\perp^{(1)}(s)\\(x_1x_2+y_1y_2)\tilde{w}_r(s)\end{pmatrix}$$

Synchronous Wave

inductive surface impedance

$$Z_s = j\omega L$$

resonance condition

$$Z_{\parallel}^{(m)} = \frac{Z_s}{2\pi R} \frac{1}{1 + j\frac{\omega}{c}\frac{R}{2}\frac{Z_s}{Z_0}}$$

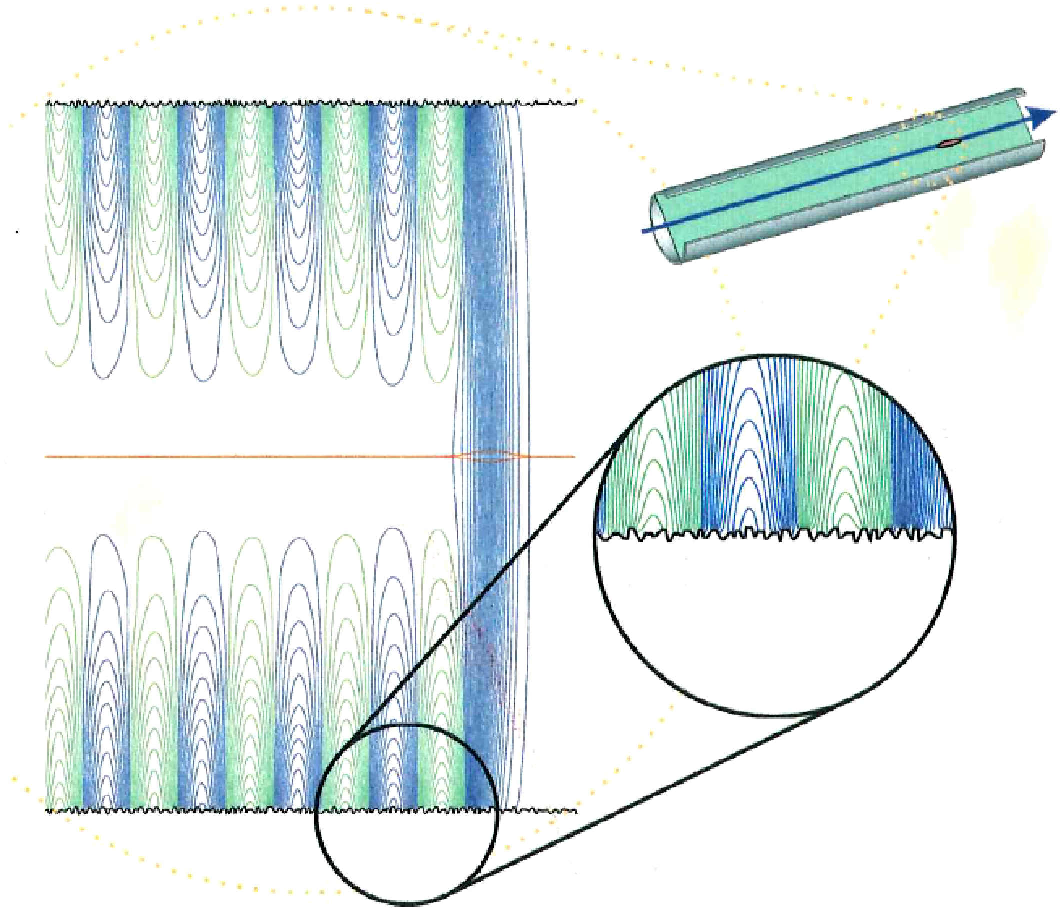
$\rightarrow 0$

$\rightarrow \omega_0$

short bunch $\frac{\sigma}{2\pi} << \frac{c}{\omega_0}$

Wake Fields of Short Ultra-Relativistic Electron Bunches

PhD Thesis Martin Bruene Timm, 2000



SASE FEL? See: Using pipe with corrugated walls for a sub-terahertz FEL

G. Stupakov, SLAC-PUB-16171, December 2014

particle source: parameter set

$q = 40\,\text{pC}$ 250E6 electrons

$\sigma_{\parallel} = 1.5\,\mu\text{m}$

$\Rightarrow \hat{I} = 3.2\,\text{kA}$

$E_{\text{av}} = 300\,\text{MeV}$

$E_{\text{rms}} = 0.6\,\text{MeV}$ slice energy spread

$\varepsilon_{\text{n}} = 0.2\,\mu\text{m}$

setup

FODO lattice: 90 deg, period = 40 cm, quadrupole length = 2cm

chicanes: length = 14 cm, magnet length = 2cm, R56 $\approx$ 11μm

6 LSCA cascades, each with 3 FODO periods, chicane in last half-period

effective free space wake

$$E_{av}(z-vt)=\frac{1}{4\pi\varepsilon_0\sigma_r^2}\int\lambda(z-vt-\xi)F\left(\frac{\xi\gamma}{\sigma_r}\right)d\xi$$

with

$$F(u)=\frac{u}{2\sqrt{2}}\int_0^\infty\frac{x\exp\left(-x^2/2\right)}{\left(x^2+u^2/2\right)^{3/2}}dx$$

or

$$F(u)=\frac{\mathrm{sgn}(u)}{2}-\frac{u\sqrt{\pi}}{4}\exp\left((u/2)^2\right)\mathrm{erfc}\left(|u/2|\right)$$

$$F(u\to 0)=\frac{\mathrm{sign}(u)}{2}$$

$$|u|>>1\Rightarrow F(u)\to\frac{\mathrm{sign}(u)}{u^2}$$