Smith-Purcell radiations for a cylindrical grating

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Outline

- Spectral-angular distribution is investigated for the radiation emitted by a point charge moving along a helical trajectory around/inside a cylindrical grating with conducting strips parallel to the cylinder axis

- In the problem under consideration two types of the radiation processes are realized: Synchrotron (Undulator) and Smith-Purcell radiations

- Primary source for both types of these emissions is the electromagnetic field of the charged particle, unlike to the synchrotron radiation the Smith-Purcell radiation is formed by the medium as a result of its dynamic polarization by the field of the moving charge.

- The effect of the grating on the radiation intensity is approximated.

- The expressions are derived for the electric and magnetic fields and for the angular density of the radiation intensity on a given harmonic.

- Smith-Purcell effect usually is considered for sources with rectilinear motion and the radiation is not superposed with other types of the radiation

Motivation

- Due to its unique characteristics, such as broad spectrum, high flux, and high degree of polarization, the synchrotron radiation is an ideal tool for many types of research and also has found industrial applications including materials science, biological and life sciences, medicine, chemistry

- These extensive applications motivate the importance of investigations for various types of mechanisms to control the radiation parameters.

- Smith-Purcell radiation arises when charged particles are in flight near a diffraction grating

- Smith-Purcell radiation presents a tunable source of the electromagnetic radiation over a wide range of frequency spectrum

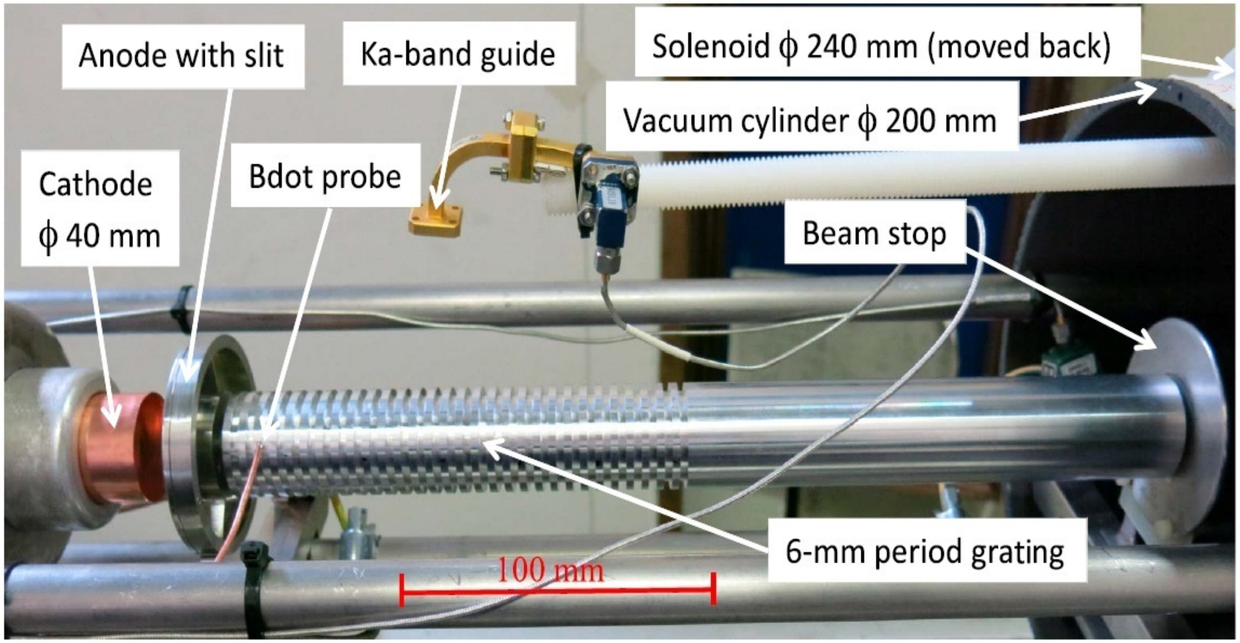
- It has a number of remarkable properties and is widely used in various fields of science and technology for the generation of electromagnetic radiation in different wavelength ranges and for the determination of the characteristics of emitting particles using the properties of the radiation field

- Smith-Purcell radiation is one of the basic mechanisms for the generation of electromagnetic waves in the millimeter and submillimeter wavelengths.

Motivation

The Smith-Purcell effect from an annular electron beam moving parallel to the axis of a cylindrical grating, with the grooves perpendicular to the axis, has been recently investigated in [1]. Of course, in this case the only radiation mechanism is the Smith-Purcell one.

[1] H.P. Bluem, R.H. Jackson, Jr., J.D. Jarvis, A.M.M. Todd, J. Gardelle, P. Modin, and J.T. Donohue, First Lasing From a High-Power Cylindrical Grating Smith–Purcell Device, IEEE Trans. Plasma Sci. 43, 3176 (2015).



Basic experimental arrangement for the cylindrical SPFEL demonstrations with the 15-GHz fundamental grating installed

Geometry of the problem

- We consider the radiation of a relativistic charged particle q, moving along a circular/helical trajectory of radius r_e inside/outside a infinitely long coaxial cylindrical grating.

- The latter consists metallic strips with width a situated on the rings with radius $r_c \lesseqgtr r_e$.

- The distance between the strips will be denoted by b.

In accordance with the problem symmetry, we use cylindrical coordinates (r, φ, z) with the axis z directed along the axis of the cylinder.

We assume that the system is immersed in a homogeneous medium with dielectric permittivity ε

Geometry of the problem

- The radius of the circular orbit and the angular frequency of motion are expressed in terms of the external magnetic field and of the charge transversal velocity by the formulas

$$r_e = \frac{m_e c v_\perp}{qH_{ext}}\gamma, \quad \omega_0 = \frac{v_\perp}{r_e} = \frac{qH_{ext}}{m_e c}$$

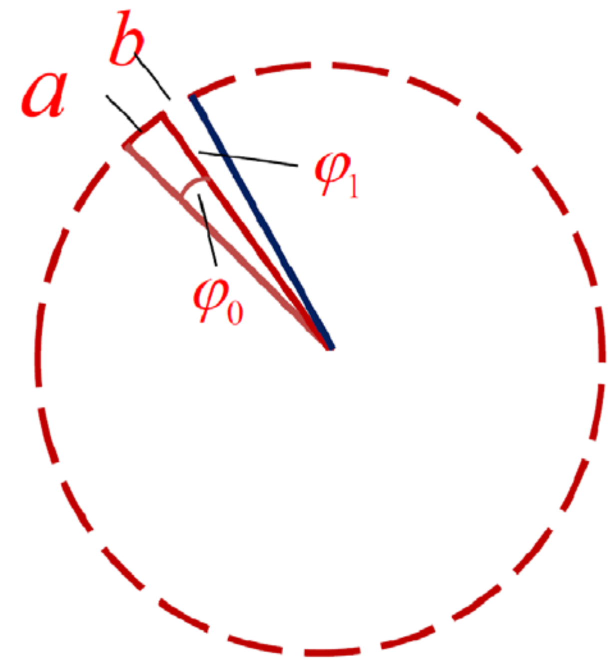
- The strips are located in the angular regions

$$\frac{2\pi m}{N} \le \varphi \le \varphi_0 + \frac{2\pi m}{N}$$

where $m = 0, 1, 2, \ldots, N$

Angular width of the strips $\qquad \varphi_0 = \frac{a}{r_e}$

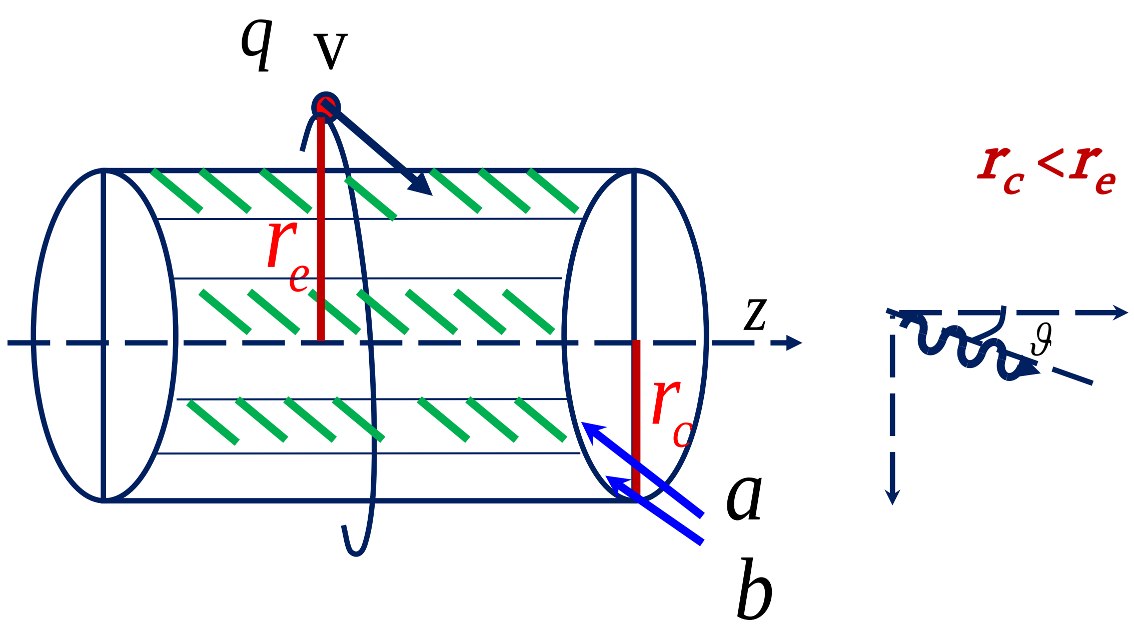
Angular period of the grating $\qquad \varphi_1 = \frac{a+b}{r_e}$



$$N = \frac{2\pi r_c}{a+b}$$

Number of periods in the grating

Geometry of the problem



$\boldsymbol{r_c < r_e}$

Method for Calculation

- The problem is mathematically complicated and an exact analytical result is not available. This is already the case for simpler problems of the Smith-Purcell radiation from planar gratings. For the latter geometry various approximate analytic and numerical methods have been developed for the evaluation of the spectral-angular distribution of the radiation intensity (for the comparison of different models in the calculations of the Smith-Purcell radiation intensity see [1]).

- The approximation by the surface currents induced on the strips by the field of the moving charge for planar gratings has been discussed in [2, 3] (for further developments of the method see [4]).

[1] D.V. Karlovets and A.P. Potylitsyn, Phys. Rev. ST Accel. Beams 9, 080701 (2006).
[2] J. Walsh, K. Woods, and S. Yeager, Nucl. Instrum. Methods Phys. Res., Sect. A 341, 277 (1994).
[3] J.H. Brownell, J. Walsh, and G. Doucas, Phys. Rev. E 57, 1075 (1998).
[4] D.V. Karlovets and A.P. Potylitsyn, Phys. Lett. A 373, 1988 (2009);
D.V. Karlovets and A.P. Potylitsyn, arXiv:0908.2336; D.V. Karlovets, JETP 113, 27 (2011).

Current density

- ✓ For the current density induced on the strips of the cylindrical grating one has

$$j_l^{(s)} = v_l' \sigma^{(s)}(\varphi,\ z,\ t)(r - r_c) \qquad v_l' = v'\delta_{2l},\quad l = 1,2,3,\ (r,\varphi,z)$$

$$v' = \omega_0 r_c$$

- ✓ For the surface charge density we use the expression

$$\sigma^{(s)}(\varphi,\ z,\ t) = \begin{cases} \sigma(\varphi,\ z,\ t), & m\varphi_1 \le \varphi \le m\varphi_1 + \varphi_0 \\ 0, & \text{otherwise} \end{cases} \qquad m = 0,\ 1,\ 2,\ \ldots, N-1,$$

Current density

For the corresponding Fourier transform, $\sigma_n\left(k_z\right)$, defined in accordance with

$$\sigma^{(s)}(\varphi,\ z,\ t)=\sum_{n=-\infty}^{+\infty} e^{in(\varphi-\omega_0 t)}\int_{-\infty}^{+\infty} dk_z e^{ik_z z}\sigma_n\left(k_z\right)$$

By using the expression for the radial component of the electric field it can be seen that

$$\sigma_n\left(k_z\right)=\frac{E_{nr,r=r_c}\left(k_z\right)}{4\pi}=-\frac{qr_e}{8\pi^2 r_c^2}\sum_{\alpha=\pm 1}\frac{H_{n+\alpha}\left(\lambda r_e\right)}{H_{n+\alpha}\left(\lambda r_c\right)} \qquad \lambda=\sqrt{\frac{n^2\omega_0^2}{c^2}-k_z^2}$$

It is easy to see that, in the limit $r_e \to r_c$ for the surface charge one gets

$\sigma^{(s)}(\varphi,\ z,\ t)_{r_e\to r_c}=-q\delta\left(\varphi-\omega_0 t\right)\delta\left(z\right)/r_c$ where we have assumed that $0\leq\varphi-\omega_0 t<2\pi$

The latter corresponds to a point charge $-q$ located on the cylinder surface

Electromagnetic fields

The vector potential for the field generated by the surface current density is determined from the formula

$$A_i^{(\mathrm{s})}(\mathbf{r},t) = -\frac{1}{2\pi^2 c}\int G_{il}(\mathbf{r},t,\mathbf{r}',t')j_l^{(\mathrm{s})}(\mathbf{r}',t')d\mathbf{r}'dt',$$

where $G_{il}(\boldsymbol{r},t,\boldsymbol{r}',t')$ is the electromagnetic field Green function in a vacuum and summation over l is understood.

In accordance with the problem symmetry, for the Green function we have the following Fourier expansion

$$G_{il}(\mathbf{r},t,\mathbf{r}',t') = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}dk_z\int_{-\infty}^{+\infty}d\omega\, G_{il}(n,k_z,\omega,r,r')e^{in(\varphi-\varphi')+ik_z(z-z')-i\omega(t-t')}$$

$$F_l(x) = F_l^{(0)}(x) + F_l^{(\mathrm{s})}(x)$$

Having the fields, we can evaluate the radiation intensity from a charge rotating around a diffraction grating. The average energy flux per unit time through the cylindrical surface of radius r, coaxial with the axis of the grating, is given by

$$I^{(g)} = \frac{c}{4\pi T}\int_0^{2\pi}d\varphi\int_0^T dt\int_{-\infty}^{\infty}dz\, r\mathbf{n}_r\cdot[\mathbf{E}\times\mathbf{H}]$$

$T = 2\pi/\omega_0$ and $\mathbf{n}_r$ is the unit vector along the radial direction r

Radiation intensity

$$I^{(g)} = 2\pi \sum_{n=1}^{\infty} \int_0^{\pi} d\theta \, \sin\theta \frac{dI_n^{(g)}}{d\Omega},$$

the angular density of the radiation intensity at a given harmonic

$$\frac{dI_n^{(g)}}{d\Omega} = \frac{q^2 \beta^2 n^2 \omega_0^2}{8\pi c\sqrt{\varepsilon}} \sum_{m=-\infty}^{+\infty} \left[\left| R_{n,m}^{(+1)} - R_{n,m}^{(-1)} \right|^2 + \cos^2\theta \left| R_{n,m}^{(+1)} + R_{n,m}^{(-1)} \right|^2 \right]$$

$$R_{n,m}^{(\alpha)} = \delta_{0m} J_{n+\alpha}(n\beta \sin\theta) - S_m \frac{H_{n+\alpha}(n\beta\sin\theta)}{H_{n+\alpha}(n\beta'\sin\theta)} J_{n+mN+\alpha}(n\beta'\sin\theta)$$

$$S_m = \frac{1}{\pi m} \sin\left(\frac{\pi m a}{a+b}\right) \qquad \beta' = r_c \beta / r_e$$

The part in the radiation intensity coming from the first term corresponds to the radiation in a homogeneous medium.

The remaining parts are induced by the diffraction grating.

Limiting cases

- $a = 0$ $$\frac{dI_n^{(0)}}{d\Omega} = \frac{q^2 n^2 \omega_0^2}{2\pi c\sqrt{\varepsilon}} \left[\beta^2 J_n'^2(n\beta\sin\theta) + \cot^2\theta J_n^2(n\beta\sin\theta)\right]$$
- $b = 0$ $$\frac{dI_n^{(c)}}{d\Omega} = \frac{q^2 n^2 \omega_0^2}{8\pi\sqrt{\varepsilon}c}\beta^2 \left[\left|B_{n+1} - B_{n-1}\right|^2 + \left|B_{n+1} + B_{n-1}\right|^2 \cos^2\theta\right]$$

$$B_{n+\alpha} = J_{n+\alpha}(\lambda r_e) - \frac{H_{n+\alpha}(\lambda r_e)}{H_{n+\alpha}(\lambda r_c)} J_{n+\alpha}(\lambda r_c).$$

Numerical analysis

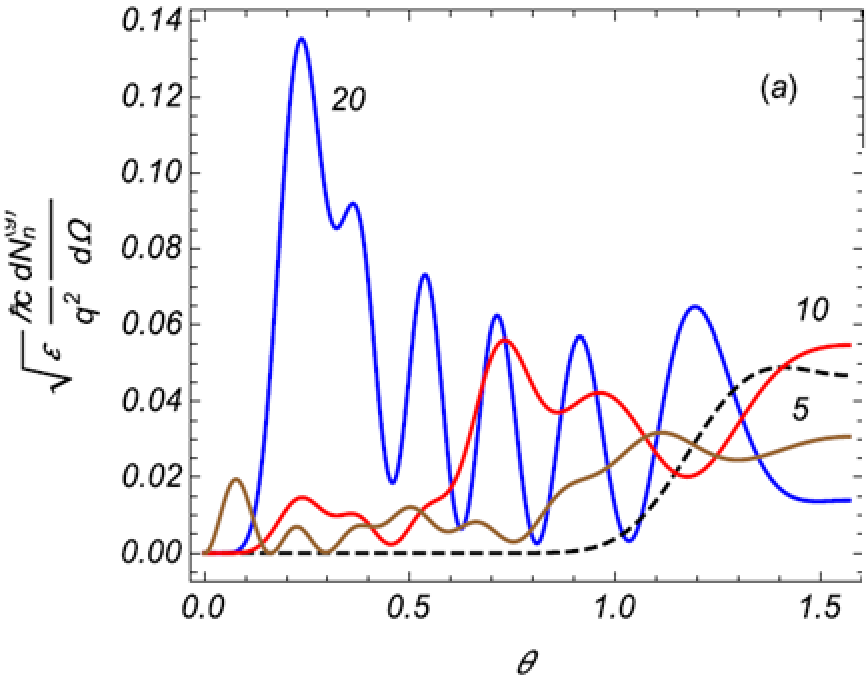
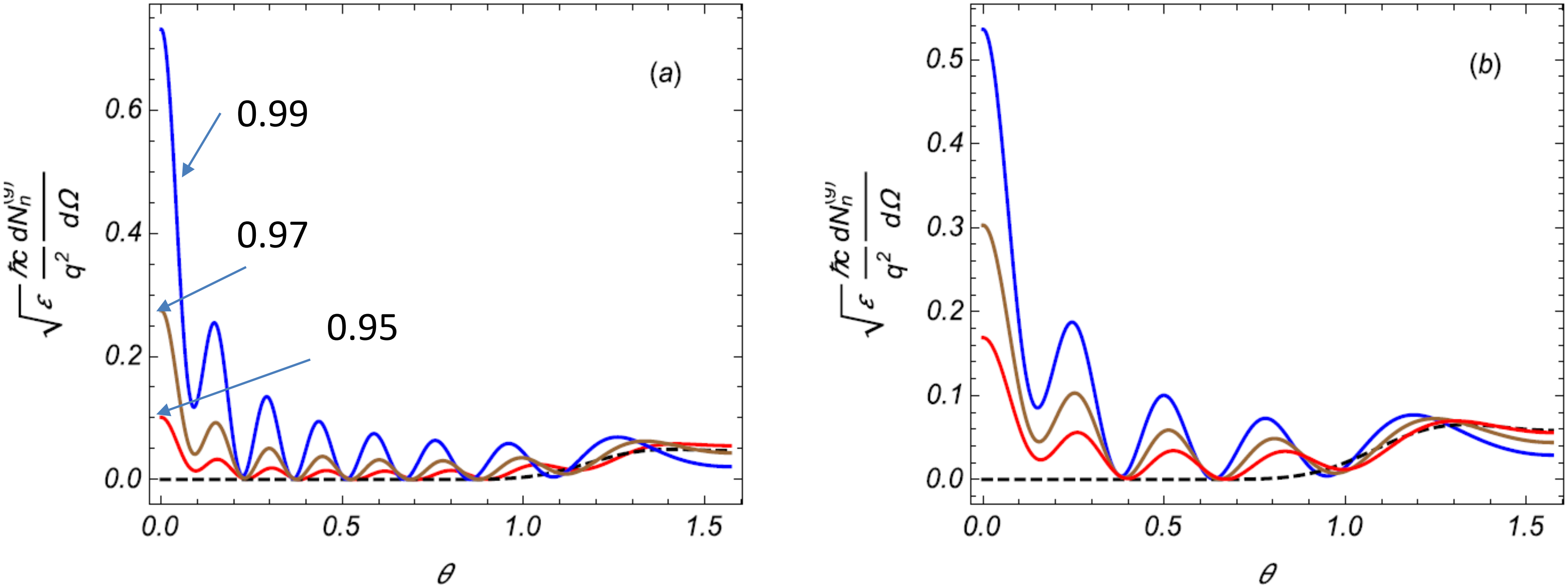


Fig. 1. The angular dependence of the number of the radiated quanta per period of the charge rotation for the electron energy 2 MeV and for the radiation harmonic $n = 25$. The numbers near the curves are the values of N. For the parameters we have taken $r_c/r_e = 0.99$, $b/a = 1$ and $\varepsilon = 1$.

For the angles not close to the rotation plane, the radiation intensity is dominated by the Smith-Purcell part.

Numerical analysis

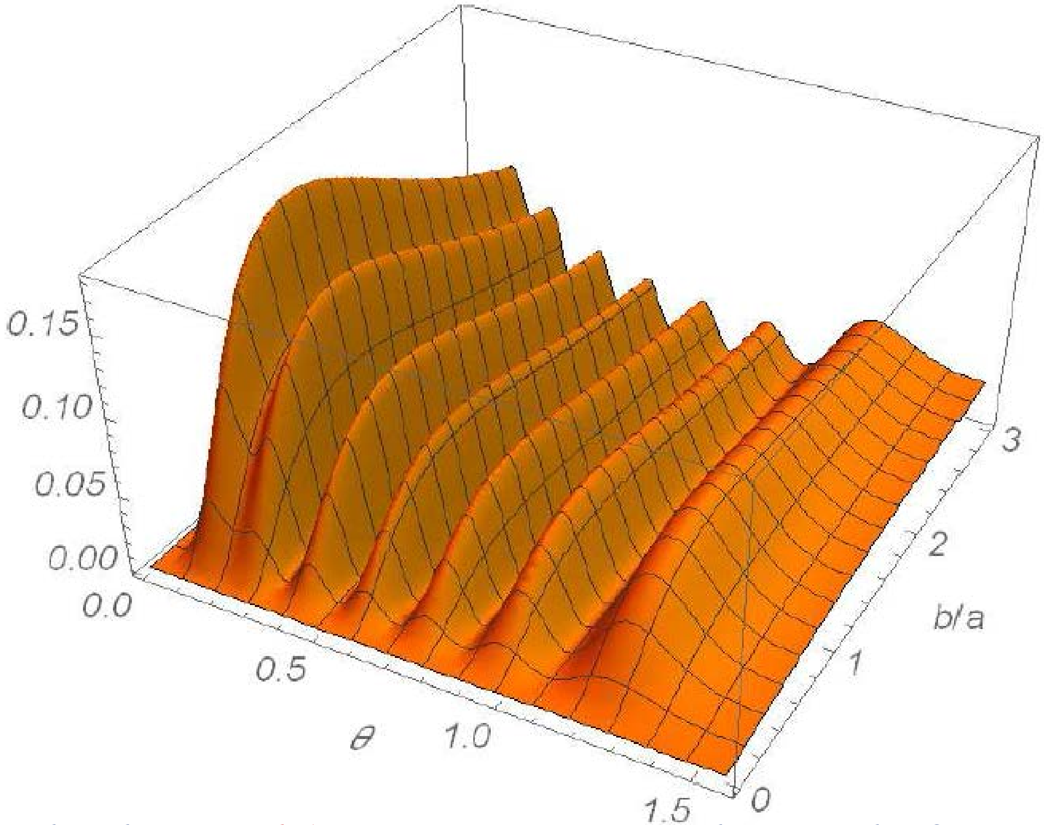


The angular density of the number of the radiated quanta as a function of θ for the electron energy 2 MeV and for $r_c/r_e = 0.95, 0.97, 0.99$ (upper curves near $\theta = 0$ correspond to larger values of r_c/r_e). The panels (a) and (b) are plotted for $n = 25$, $N = 24$ and $n = 15$, $N = 14$, respectively.

For a given harmonic n, the strongest radiation at small angles θ is obtained for the number of periods $N = n \pm 1$.

For given characteristics of the charge (energy, radius of the orbit), by the choice of the parameters for the diffraction grating, one can have highly directional radiation on a given harmonic n directed near the normal to the plane of the charge rotation.

Numerical analysis



Number of the radiated quanta on the harmonic *n= 25* versus θ and the ratio *b/a* for the electron with the energy *2 MeV*.

For the values of the other parameters we have taken $N = 28$, $r_c / r_e = 0.99$.

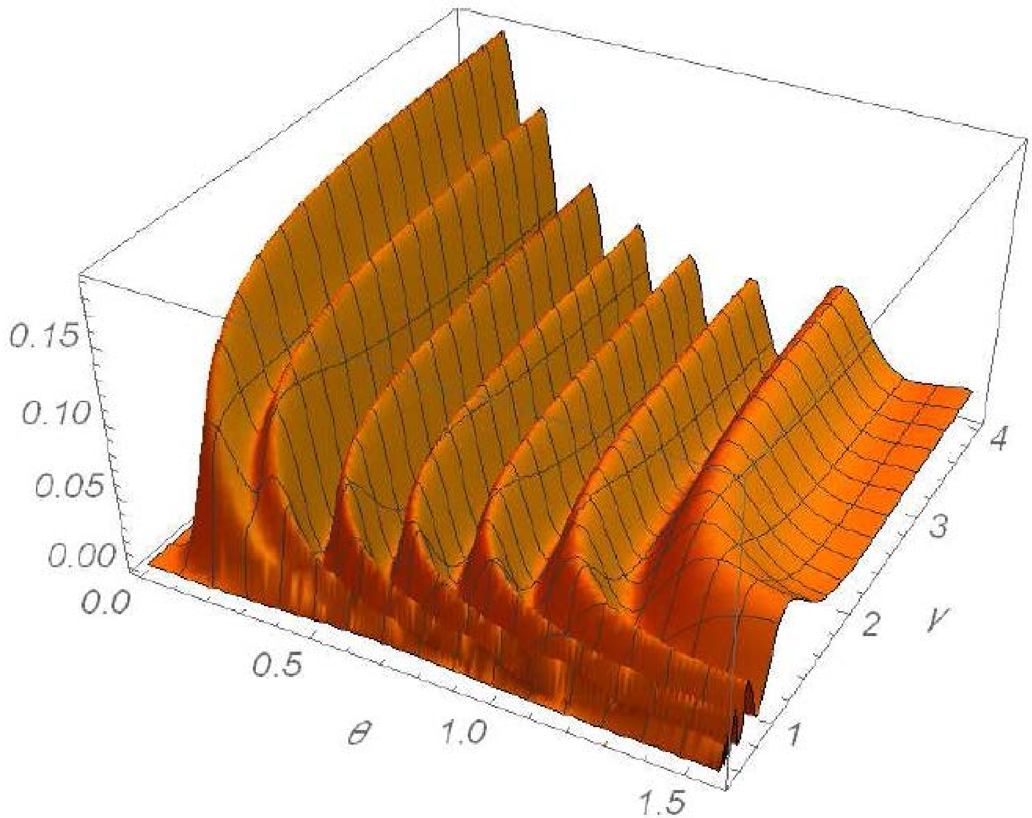
In the limit *b/a≪1* we recover the result for a charge rotating around a conducting cylinder.

In the opposite limit *b/a≫1* the result for the rotation in the vacuum is obtained.

Locations of the angular peaks are not sensitive to the ratio *b/a.*

As a function of *b/a*, the radiation intensity takes its maximum value for *b/a* close to *1*.

Numerical analysis

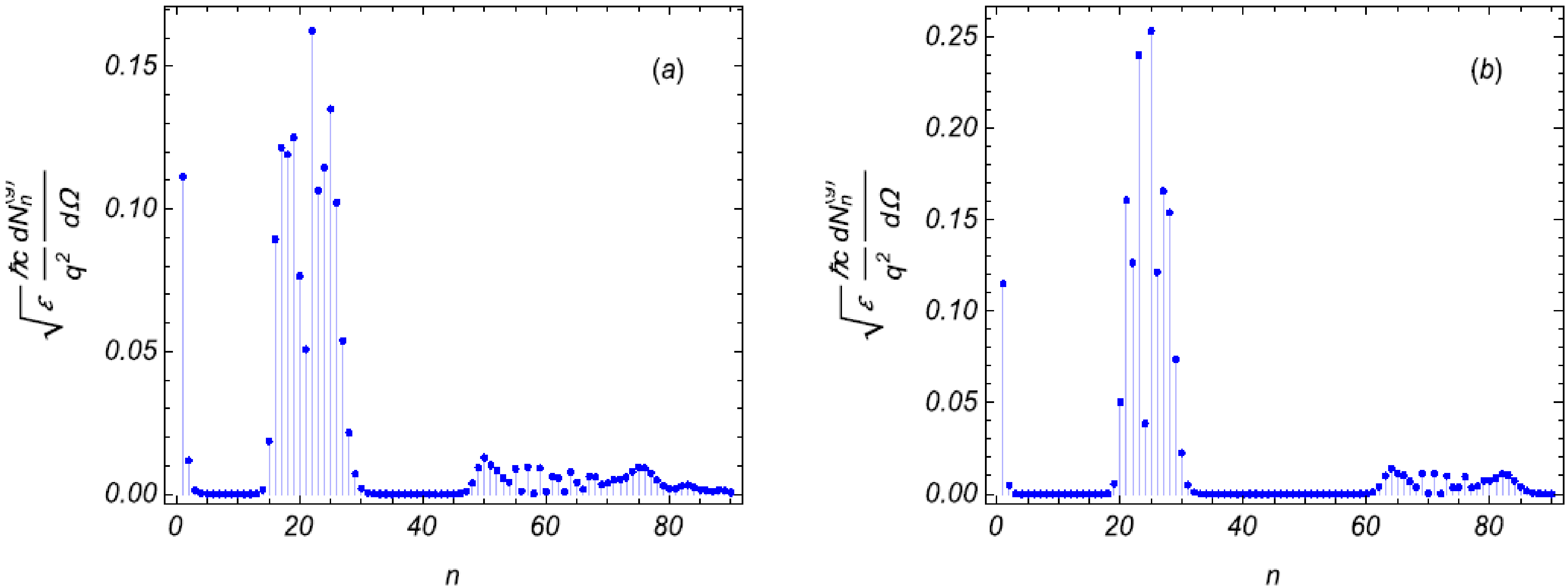


Number of the radiated quanta on the harmonic $n = 25$ versus θ and the $\gamma = E_e/(m_e c^2)$ for $b/a = 1$, with E_e being the energy of the electron.

For the values of the other parameters we have taken $N = 28$, $r_c/r_e = 0.99$.

For $\gamma \gtrsim 3$ and for the angles θ not to close to $\pi/2$ (rotation plane), the heights and the locations of the angular peaks in the radiation intensity are not sensitive to the value of the electron energy.

Numerical analysis



The angular density of the number of the radiated quanta as a function of the radiation harmonic for the electron energy 2 MeV and for $r_c/r_e = 0.99$. For the panels (a) and (b) one has $N = 20$, $\theta = 0.24$ and $N = 24$, $\theta = 0.15$, respectively.

These values of the angle correspond to the local peaks in the angular distribution for the examples presented in figures above. For these angles the radiation is mainly dominated by the Smith-Purcell part.

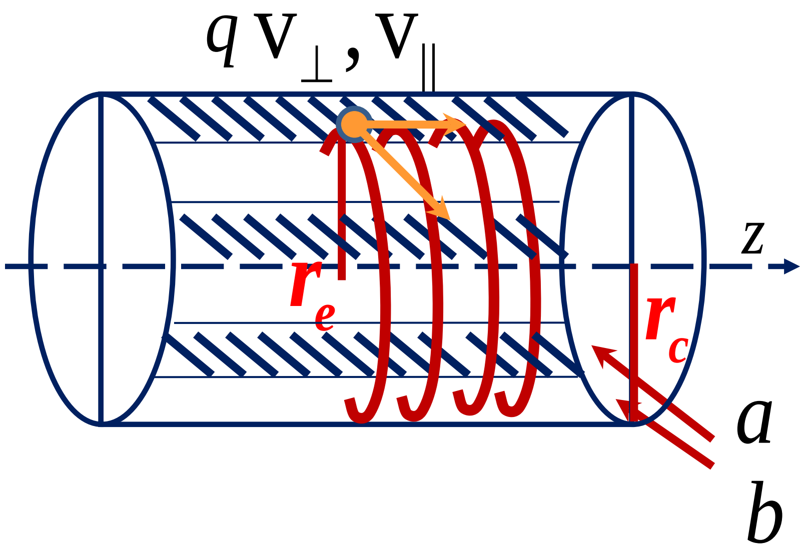
Synchrotron part of the radiation is mainly located near the angle $\theta = \pi/2$

Corresponding radiation intensity, increases with increasing *n* up to the values $n_{\max} \approx \gamma^3$

For the energy corresponding to figure one has $n_{\max} \approx 60.$

As regards the angular density of the number of the radiated quanta for synchrotron radiation monotonically decreases with increasing *n*

Geometry of the problem



$r_c > r_e$

Current density

We will assume that the charge trajectory is sufficiently close to the grating and will approximate the radiation field by the field from the source

Region between the strips

$$j_l = \begin{cases} \dfrac{q}{r} \mathrm{v}_l \delta(r - r_e)\delta(\varphi - \omega_0 t)\delta(z - v_{\parallel} t), & \text{for} \quad m\varphi_0 + m\varphi_1 \le \varphi \le (m+1)\varphi_1 \\ 0, & \text{for} \quad m\varphi_1 \le \varphi \le m\varphi_1 + \varphi_0 \end{cases}$$

m = 0, 1, 2, …,N – 1,

Locations of the strips

$$v_r = 0$$

$$v_\varphi = \omega_0 r_e$$

$$v_z = v_{\parallel}$$

This corresponds to rectangular oscillations of the charge

Approximation allows us to obtain closed expressions for the electromagnetic fields and radiation intensity in the region outside the grating

Electromagnetic fields

Having the current density, the Fourier components of vector potential is expressed as

$$A_{nml}(k_z,r) = \frac{i^{1-l}qv_\perp}{4c}s_m\sum_{\alpha=\pm1}\alpha^l J_{n+Nm+\alpha}(\lambda r_e)H_{n+Nm+\alpha}(\lambda r),$$

$$A_{nm3}(k_z,r) = \frac{qv_\parallel}{2ic}s_m J_{n+Nm}\left(\lambda r_e\right)H_{n+Nm}\left(\lambda r\right),$$

$$\boxed{s_m = \begin{cases} \frac{1}{\pi m}e^{-imN\varphi_0/2}\sin\left(mN\varphi_0/2\right), & m\neq 0,\\ N\varphi_0/2\pi - 1, & m=0.\end{cases}}$$

$$\boxed{\lambda = \sqrt{\omega_n^2(k_z)\varepsilon/c^2 - k_z^2}} \quad \boxed{\omega_n(k_z) = n\omega_0 + k_z v_\parallel}$$

Radiation intensity

Angular density of the radiation intensity on a given harmonic

$$\frac{dI_n}{d\Omega} = \frac{q^2 n^2 \omega_0^2}{2\pi c\sqrt{\varepsilon}\left|1-\beta_{||}\cos\theta\right|^3}\sum_{m=-\infty}^{+\infty}|s_m|^2\left\{\beta_\perp^2 J'^2_{n+Nm}(nu(\theta))\right.$$

$$\left.+\left[\cos\theta\left(1+\left(1-\beta_{||}\cos\theta\right)Nm/n\right)-\beta_{||}\right]^2\frac{J^2_{n+Nm}\left(nu(\theta)\right)}{\sin^2\theta}\right\}$$

$$d\Omega = \sin\theta d\theta d\phi \qquad u(\theta) = \frac{\beta_\perp\sin\theta}{1-\beta_{||}\cos\theta}.$$

The spectrum of radiation

For the radiation at $n \neq 0$ harmonics, the allowed values for k_z are determined from the inequality $\lambda^2 > 0$. The latter is rewritten as

$$k_z^2(1-\beta_{||}^{-2}) + 2k_z n\omega_0/v_{||} + (n\omega_0/v_{||})^2 > 0$$

Introducing a new angular variable θ, $0 \leqslant \theta \leqslant \pi$, the solution of this inequality is presented as

$$k_z = \frac{n\omega_0}{c}\frac{\sqrt{\varepsilon}\cos\theta}{1-\beta_{||}\cos\theta}$$

$$\omega_n(\theta) = |\omega_n(k_z)| = \frac{n\omega_0}{|1-\beta_{||}\cos\theta|}, \quad n = 1, 2, \ldots,$$

This corresponds to wave with frequency propagating at the angle ϑ with respect to the z-axis. Formula describes the normal Doppler effect in the cases $\beta_{||} < 1$ and $\beta_{||} > 1$, $\vartheta > \vartheta_C$, $\vartheta_C = \arccos(1/\beta_{||})$, and anomalous Doppler effect inside the Cherenkov cone, $\vartheta < \vartheta_C$ in the case $\beta_{||} > 1$.

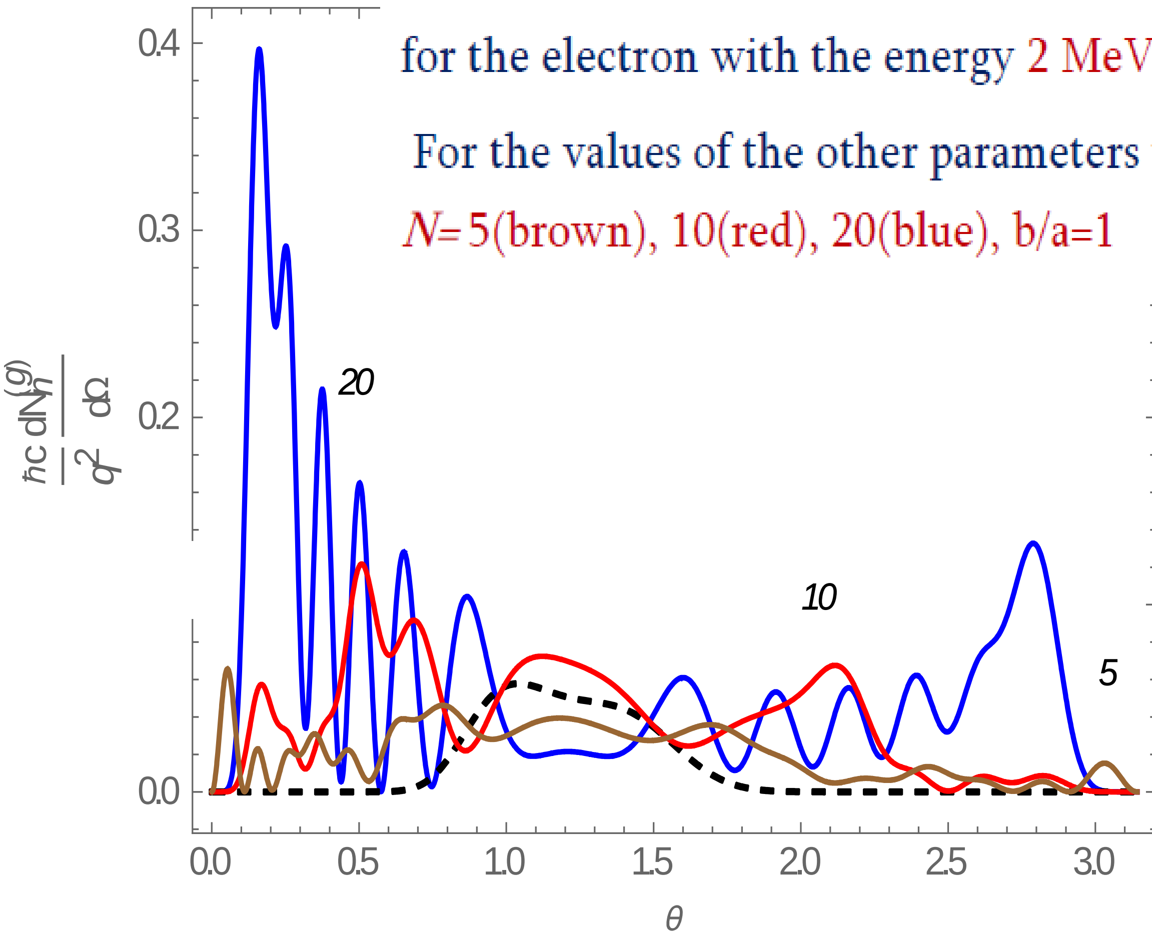
Numerical analysis

Number of the radiated quanta on the harmonic *n=25* versus θ

for the electron with the energy 2 MeV, ($\beta_{\parallel} = 0.35, \quad \beta_{\perp} = 0.9$)

For the values of the other parameters we have taken

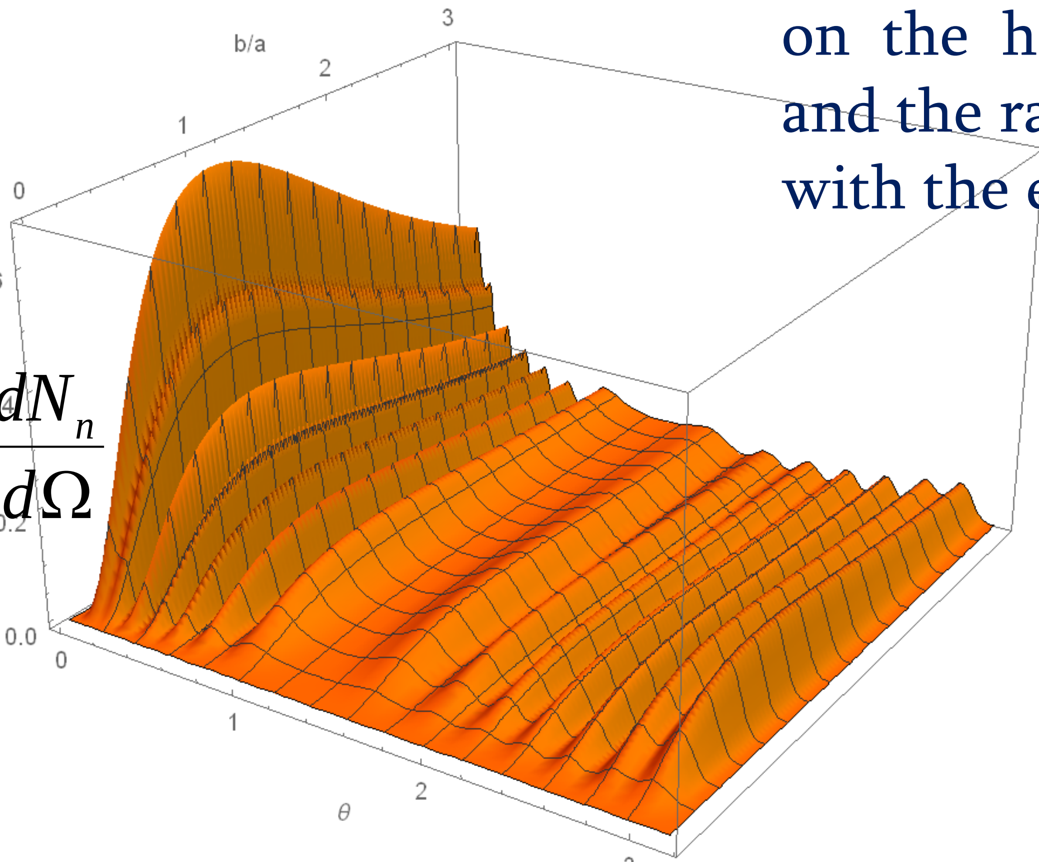
N= 5(brown), 10(red), 20(blue), b/a=1



For the angles not close to the rotation plane, the radiation intensity is dominated by the Smith-Purcell part

Numerical analysis

Number of the radiated quanta on the harmonic *n= 25* versus θ and the ratio *b/a* for the electron with the energy $2\,MeV$, $N = 28$

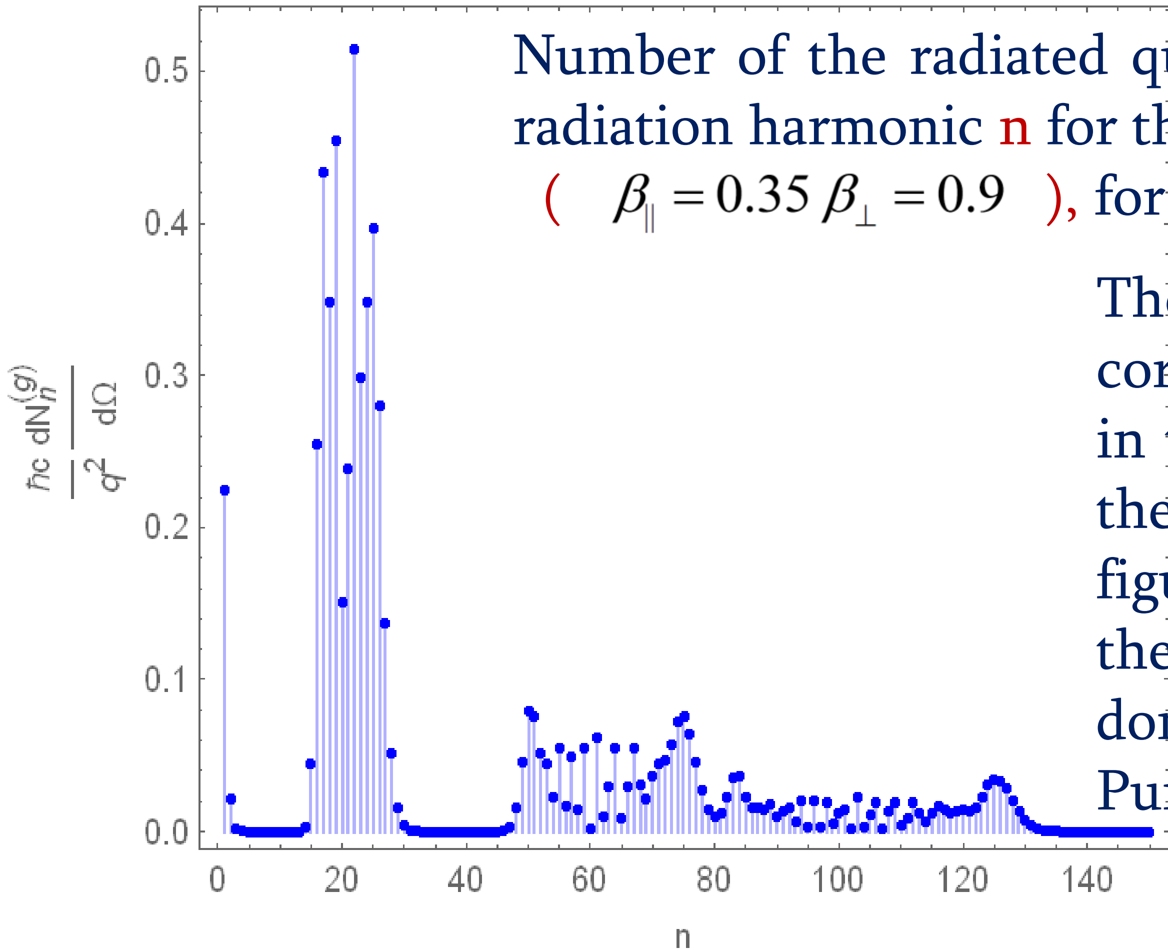


$$\frac{\hbar c}{q^2}\frac{dN_n}{d\Omega}$$

Locations of the angular peaks are not sensitive to the ratio *b/a.*

As a function of *b/a*, the radiation intensity takes its maximum value for *b/a* close to *1.*

Numerical analysis



Number of the radiated quanta as a function of the radiation harmonic n for the electron energy 2 MeV, ($\beta_{\parallel} = 0.35 \ \beta_{\perp} = 0.9$), for N=20, b/a=1. $\theta = 0.16$

These value of the angle corresponds to the local peak in the angular distribution for the examples presented in figure above. For this angle the radiation is mainly dominated by the Smith-Purcell part.

Synchrotron part of the radiation is mainly located near the angle $\theta = \pi/2$

Estimates

Maximal magnetic fields available in laboratory $H \approx 45T$ which corresponds to $H \approx 4.5\times10^5 G$. For the radius of the circular orbit one has ($1\ MeV = 1.6\times10^{-6}$ erg)

$$r = \frac{\beta E_e}{eH} = \frac{1.6\times10^{-6}}{4.8\times10^{-10}} \frac{\beta E_e(MeV)}{H(G)} cm = 3.3\times10^3 \frac{\beta E_e(MeV)}{H(G)} cm$$

For the electron energy 2MeV for the magnetic field $H \approx 5\times10^3 G$ the radius of the orbit is ≈ 1 cm. For ω_0 one has $\omega_0 = v_\perp / r_0$.

For $v_\perp \approx c$, $\omega_0 \approx 3\times10^{10}$Hz

For the radiation frequency one has $\omega_n(\theta) = \dfrac{n\omega_0}{1-\beta_{\|}\cos\theta}, \; n = 1, 2, \ldots,$

Two cases should be considered separately

Estimates

✦If the angle is not close to θ=0, then for the radiation frequency

$$\nu_n = \frac{\omega_n}{2\pi} \approx n\frac{\omega_0}{2\pi}$$

For the radiation harmonic n of the order 100 the radiation frequency is in the terahertz range, $\nu \sim 10^{12}$Hz

✦In the second case for the relativistic longitudinal motion and for the radiation along small angles θ (this case is realized in helical undulators) one has

$$1-\beta_{\|}\cos\theta \approx 1-\beta_{\|}+\beta_{\|}\theta^2/2 \approx \left(\gamma_{\|}^{2}+\theta^2\right)/2 \qquad \omega_n = \frac{2n\omega_0\gamma_{\|}^2}{1+\gamma_{\|}^2\theta^2}$$

In this case the radiation is mainly along the angles $\theta \sim 1/\gamma_{\|}$ and the radiation frequency is in the range

$$\nu_n = \frac{\omega_n}{2\pi} \approx \frac{n\omega_0}{2\pi}\gamma_{\|}^2$$

For the longitudinal energy of the order 10MeV one has $\gamma_{\|} \approx 20$ and the radiation is in the optical range

Radiation in helical undulators

For helical undulators it is introduced the parameter K_u

$$\beta_\perp = \frac{K_u}{\gamma},\ \beta_\| = \sqrt{1 - \frac{1+K_u^2}{\gamma^2}},\ \omega_0 = ck_u\beta_\|,\ r_0 = \frac{K_u}{\gamma k_u \beta_\|}$$

For helical undulators $K_u < 1$

For $0 < 1 - \beta_\| \ll 1$ and for small angles one has

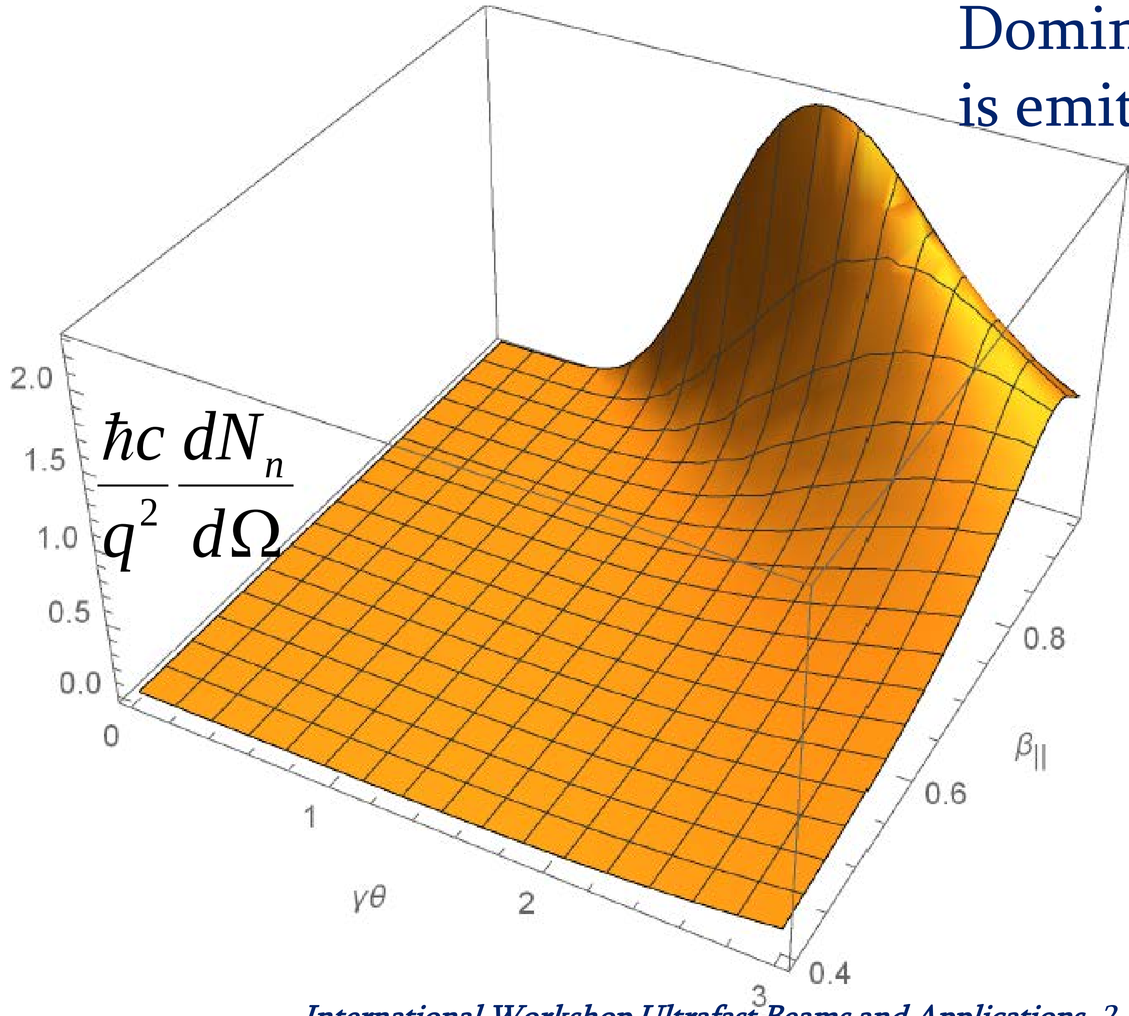
$$\omega_n = \frac{2n\omega_0\gamma_\|^2}{1+\gamma_\|^2\theta^2} \quad u \approx \frac{2K_u\theta\gamma}{1+K_u^2+\gamma^2\theta^2}$$

$$\frac{dI_n}{d\Omega} = \frac{4q^2n^2\omega_0^2\gamma^4}{\pi c\left(1+K_u^2+\gamma^2\theta^2\right)^3} \sum_{m=-\infty}^{+\infty} |s_m|^2 \left[K_u^2 J'^2_{n+Nm}(nu)\right.$$
$$\left. + \frac{\gamma^2\theta^2}{4}\left(\left(1+K_u^2+\gamma^2\theta^2\right)\frac{Nm}{n} + 1 + K_u^2 - \gamma^2\theta^2\right)^2 J^2_{n+Nm}(nu)\right]$$

Radiation is peaked along the $\theta \sim 1/\gamma$ angles

Numerical example: Pure undulator radiation

Number of the radiated quanta on the harmonic *n=5* versus θ and $\beta_{\|}$ for the $E_e = 5\ \mathrm{MeV}, K_u = 0.7$



Dominant part of the radiation is emitted along small angles

$$\theta \sim 1/\gamma$$

Undulator radiation in the presence of grating

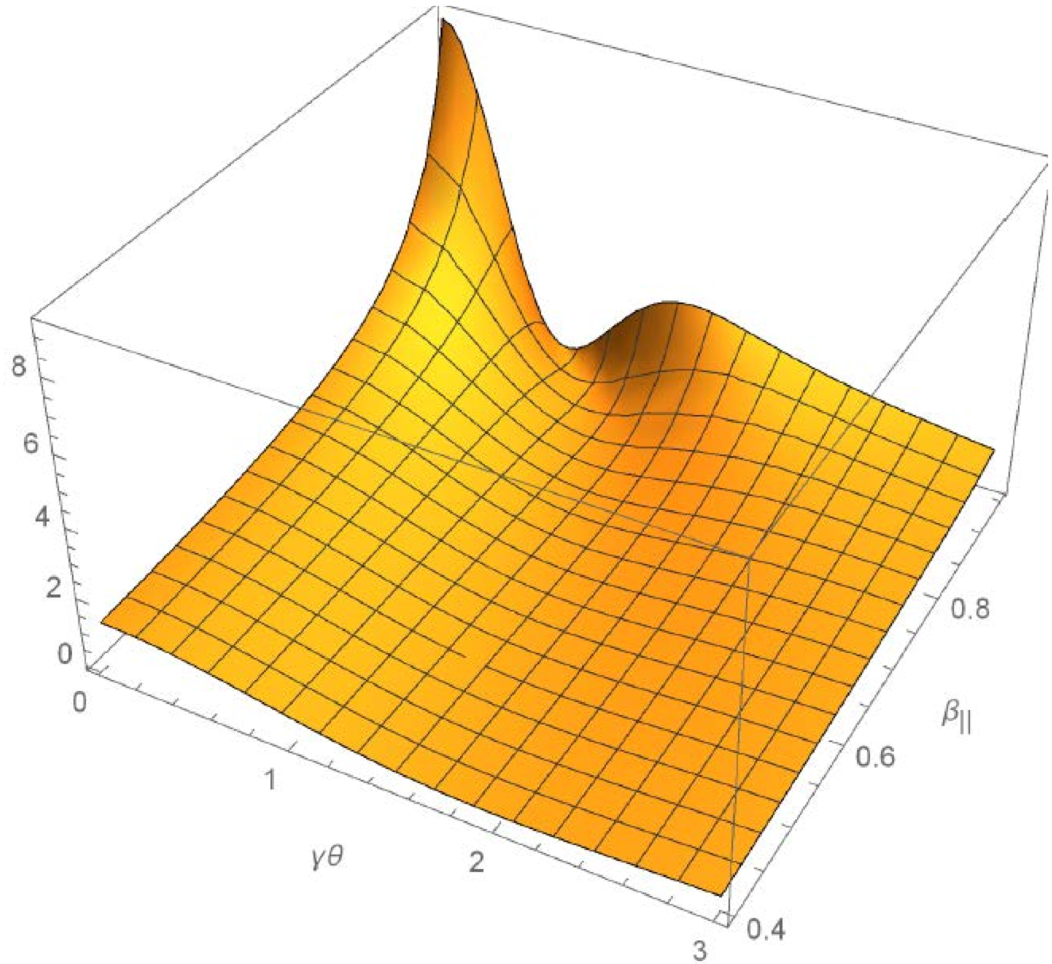
Number of the radiated quanta on the harmonic *n=5* versus θ and $\beta_{\|}$ for the $E_e = 5\ \mathrm{MeV}$

$$N = 4,\ b = a$$

$$K_u = 0.7$$

Presence of grating gives rise an additional peak at small angles

$$\frac{\hbar c}{q^2}\frac{dN_n}{d\Omega}$$



Summary

- Interference between the synchrotron and Smith-Purcell radiations may lead to interesting features

- Behavior of the radiation intensity on large harmonics can be essentially different from that for a charge rotating in the vacuum or around a solid cylinder

- Unlike to these limiting cases, for the geometry of diffraction grating the radiation intensity on higher harmonics does not vanish for small angles with respect to the cylinder axis.

- For given characteristics of the charge, by the choice of the parameters of the diffraction grating, one can have highly directional radiation near the normal to the plane of the charge rotation

- With decreasing energy, the relative contribution of the synchrotron radiation decreases and the Smith-Purcell part is dominant.