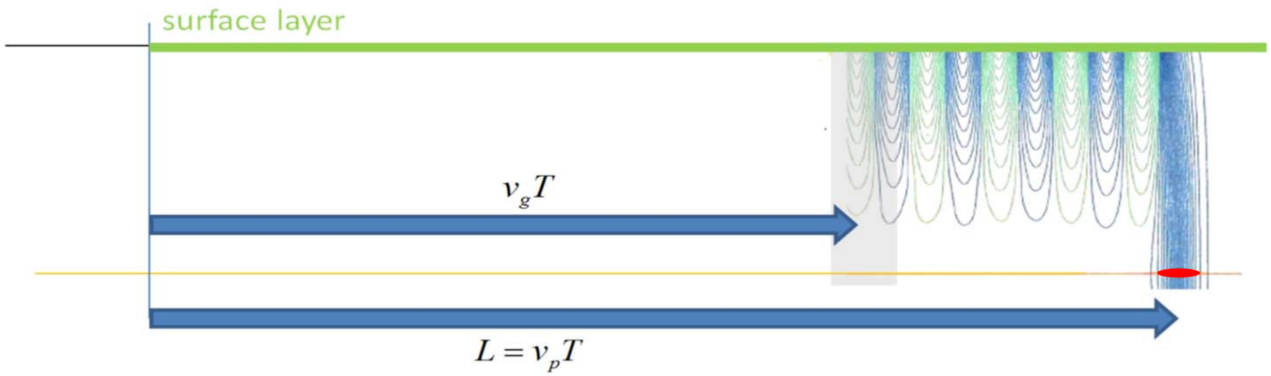
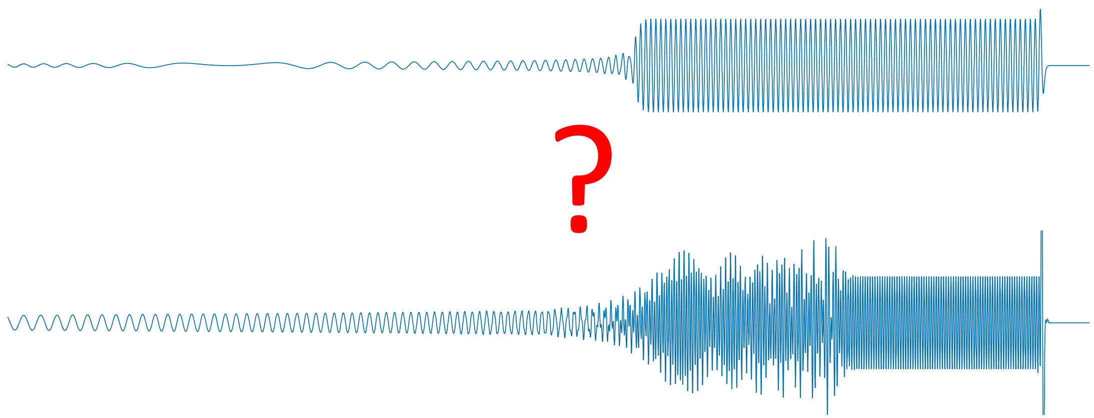
Transient Wave in a Pipe with Dielectric Layer

Martin Dohlus,
Yerevan, 3. July 2019

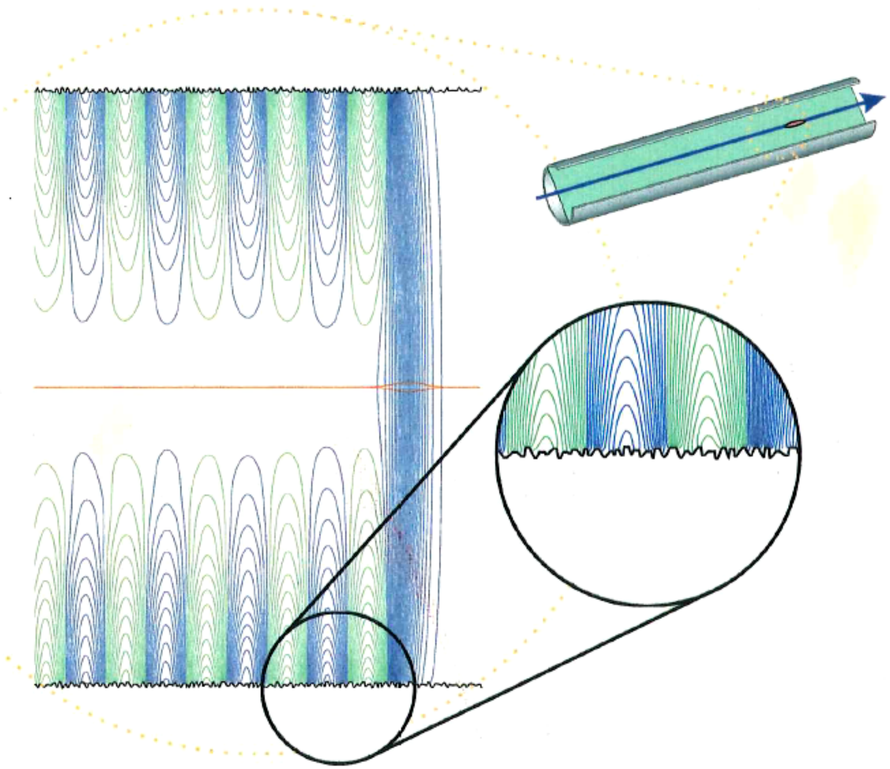




Round Pipe with Dielectric Layer

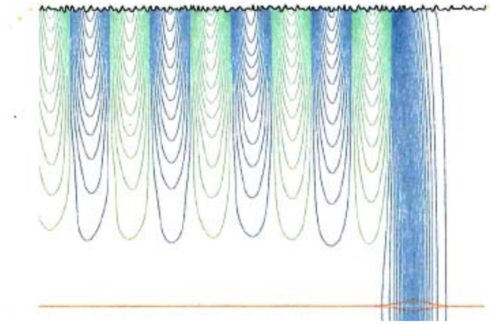
Martin Brüne Timm

Wake Fields of Short
Ultra-Relativistic Electron Bunches



or similar things (here: “artificial dielectric”)

Analytical and/or Numerical Models (by far not complete)



time domain, wake field codes

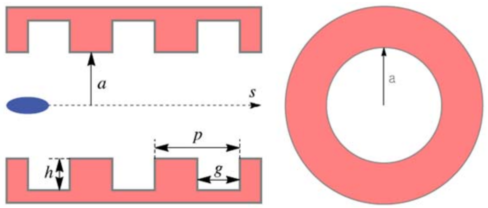
very powerful tools, in particular rz
with particles (if required), f.i. SASE effect

frequency domain, FEM

f.i.: Vlasov antenna

Stupakov: Using pipe with corrugated walls for a subterahertz free electron laser
Phys. Rev. Accel. Beams 18, 030709 (2015)

$$w(s,z) = \begin{cases} 2\kappa \cos(\omega_r z/c), & \text{for } -s(1 - v_g/v) < z < 0, \\ \kappa, & \text{for } z = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$



analytic approaches

geometry with symmetry of revolution, but not uniform in z-direction

geometry with symmetry of revolution and uniform in z-direction

Fourier method (in particular for steady state)
cavity-eigenmode-method (transient)

Fourier Method (geometry is uniform in z-direction)

curl-curl equation

$$\nabla\times\nabla\times\mathbf{E}+\mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{E}=-\frac{\partial}{\partial t}\mu\mathbf{J}$$

stimulating charge density

$$\rho(\mathbf{r},t)=\lambda(z,t)\frac{\delta(r-r_s)}{2\pi r_s}$$

2d Fourier transformation

$$\tilde{f}(k_z,\omega)=\int f(z,t)\exp\left(-j(\omega t-k_z z)\right)dtdz$$

$$(\nabla_\perp-jk_z\mathbf{e}_z)\times(\nabla_\perp-jk_z\mathbf{e}_z)\times\tilde{\mathbf{E}}-\frac{\mu\varepsilon}{\omega^2}\tilde{\mathbf{E}}=-j\omega\mu\tilde{\mathbf{J}}$$

$$\tilde{\mathbf{J}}=\mathbf{e}_z\tilde{\lambda}(k_z,\omega)\frac{\omega}{k_z}\frac{\delta(r-r_s)}{2\pi r_s}$$

(from continuity equation)

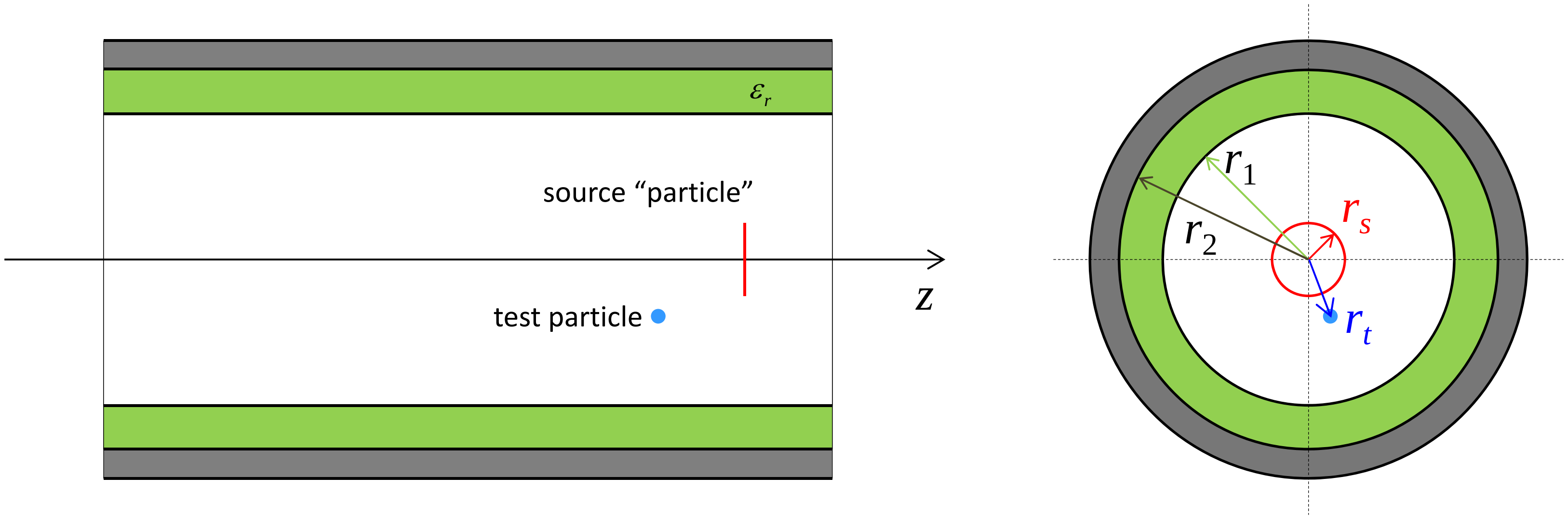
for symmetry of revolution and monopole modes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{E}_z}{\partial r}\right)+\left(\mu\varepsilon\omega^2-k_z^2\right)\tilde{E}_z=\frac{j\tilde{\rho}}{\varepsilon_0}\left(\frac{\omega^2}{c^2}\frac{1}{k_z}-k_z\right)$$

has a simple solution for layered problems

but we need a 2d inverse Fourier transformation!

my example:




```
r1     = 0.00045
r2     = 0.00055
eps_r  = 3.8
rs     = 0.0002
rt     = 0.0001
beta   = 0.99
```

the synchronous frequency is about 300 GHz

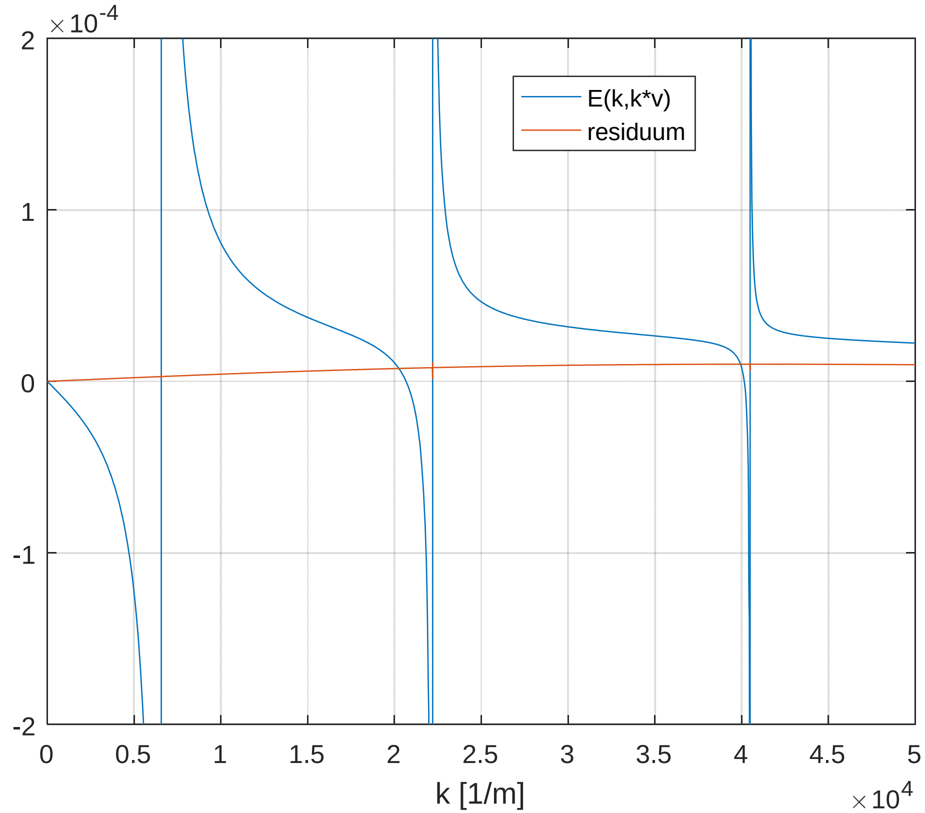
reference field (for normalization) $k_0 = \frac{q_s}{4\pi r_1^2}$

special case: rigid bunches $\lambda(z,t) = \lambda(z - vt)$

$$E_z(z,t) = E_z(z - vt) = \frac{1}{2\pi}\int \tilde{E}_z(k)\exp(-jk(z - vt))\,dk$$

it is reduced to a **1d impedance** problem

$$\mathrm{Im}\left\{\tilde{E}_z(k)/k_0\right\}$$



the function is imaginary (where it is finite)

$$\tilde{E}_z(k) = \sum_{\nu=1}^{3} A_\nu \frac{jk}{k_\nu^2 - k^2} + \boxed{rest(k)}$$

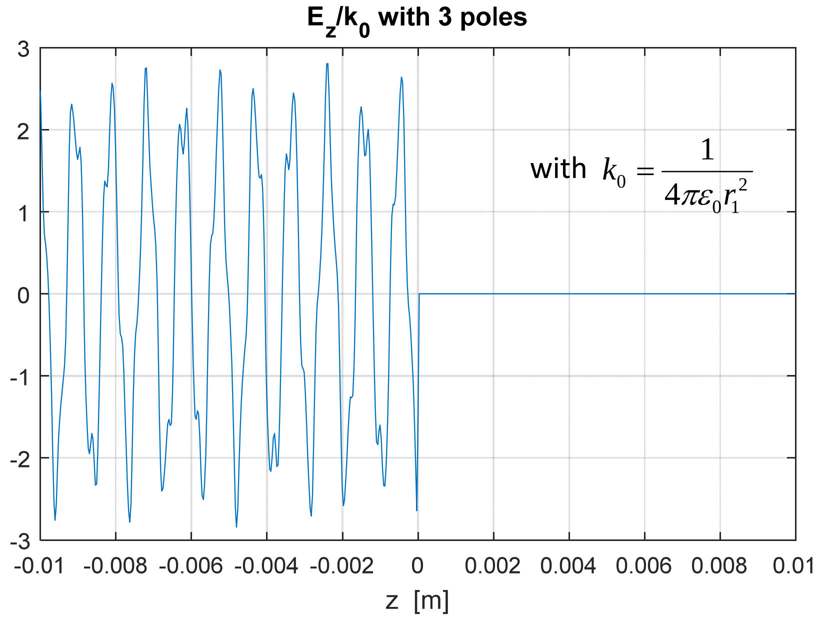
```
k1 = 6.5657e+03    A1 = -1.0114e+17
k2 = 2.2196e+04    A2 = -2.2478e+16
k3 = 4.0489e+04    A3 = -3.8157e+15
```

time domain

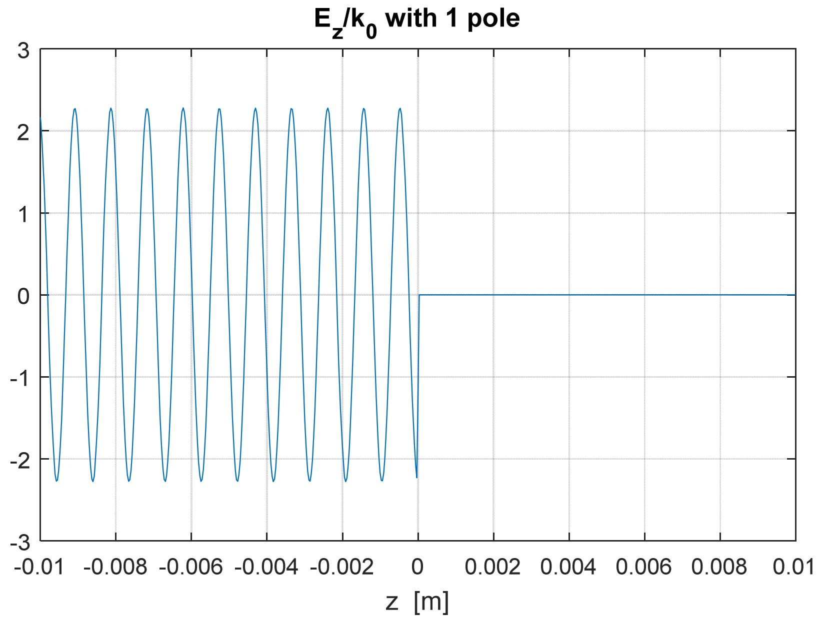
$$E_z(z)=\frac{1}{2\pi}\int \tilde{E}_z(k)\exp(-jkz)\,dk$$

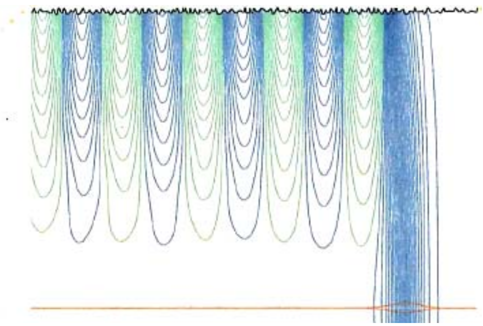
$$E_z(z<0)=\sum_{\nu=1}^{3}A_\nu\cos(k_\nu z)+res(z)$$

but the oscillating tail is of infinite length



without *res*(*z*)!

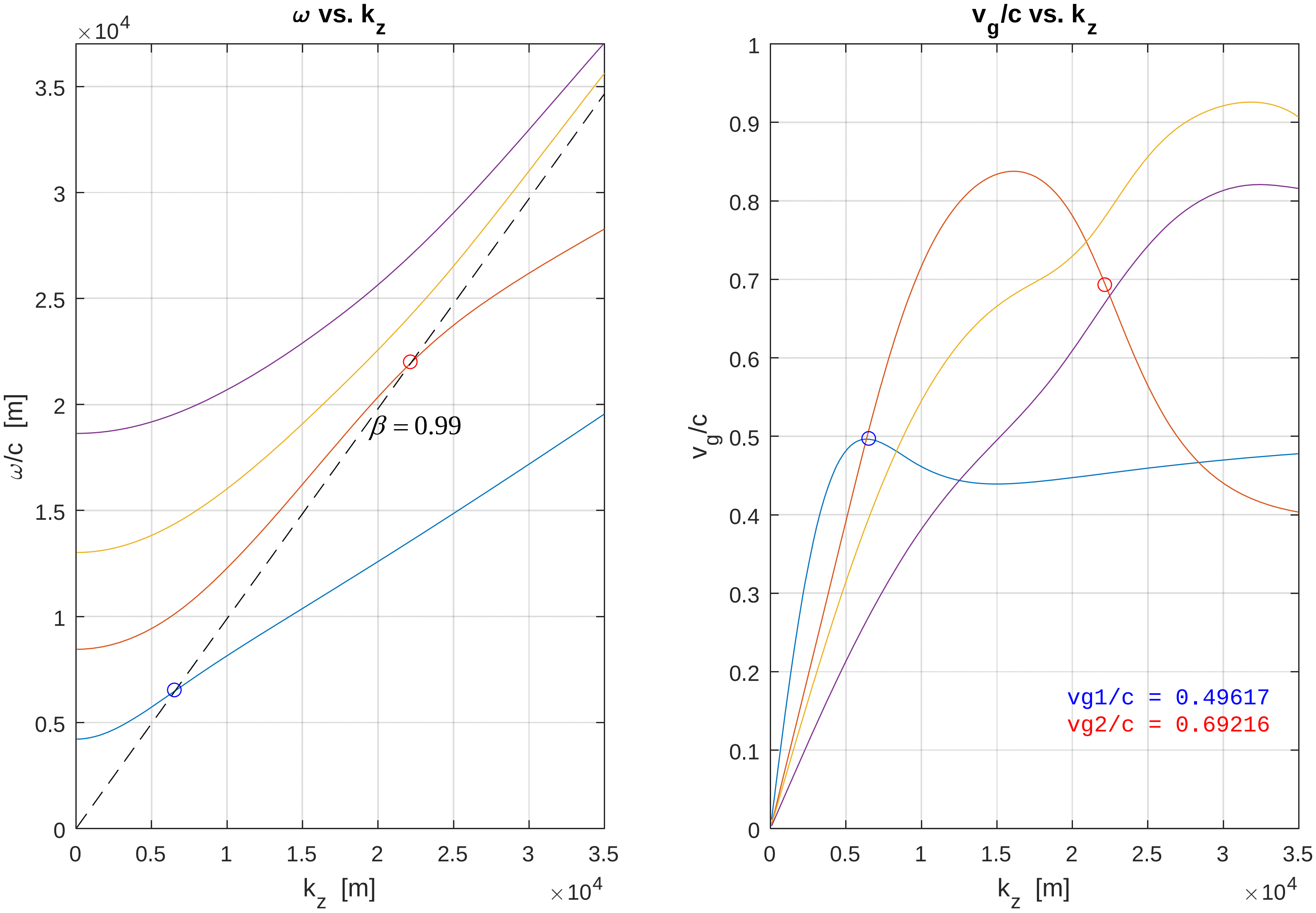




Brillouin diagram and group velocity

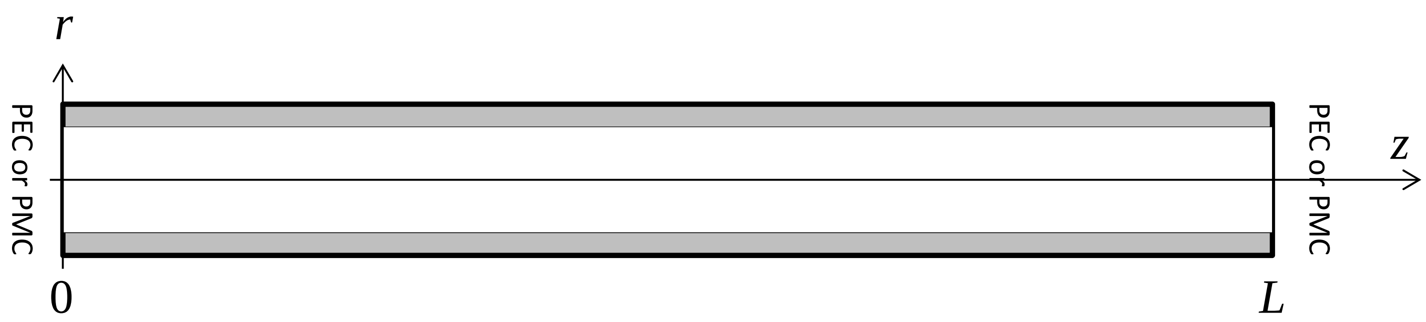
(for the first 4 monopole bands)

```
r1     = 0.00045
r2     = 0.00055
eps_r = 3.8
```



Cavity-Eigenmode-Method

we use the modes of a closed resonator system



dynamic eigenmodes related to the 1st monopole passband

$$\mathbf{E}_\nu(r,z,t) = \{\ \mathbf{e}_r E_{\nu r}(r)\sin(k_\nu z) + \mathbf{e}_z E_{\nu z}(r)\cos(k_\nu z)\}\cos(\omega_\nu t)$$ PEC boundaries

$$\mathbf{E}_\nu(r,z,t) = \{-\mathbf{e}_r E_{\nu r}(r)\cos(k_\nu z) + \mathbf{e}_z E_{\nu z}(r)\sin(k_\nu z)\}\cos(\omega_\nu t)$$ PMC boundaries

with $k_\nu = \nu \frac{\pi}{L}$

what we need to know is $\omega(k)$, $E_z = E_z(r,k)$ and the transverse component $E_r(r,k)$

this is just the solution of the waveguide problem with $\omega(k)$ the dispersion relation field amplitudes $\mathbf{E}_\nu(r,z,t) = \mathbf{E}_\nu(r,z)\cos(\omega_\nu t)$

---

curl-curl equation $\nabla\times\nabla\times\mathbf{E} + \mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{E} = -\frac{\partial}{\partial t}\mu\mathbf{J}$

current stimulation $\mathbf{J}(\mathbf{r},t) = qv\mathbf{e}_z\delta(z-vt)\frac{\delta(r-r_s)}{2\pi r_s}$

eigenmodes $\nabla\times\nabla\times\mathbf{E}_\nu = \mu\varepsilon\omega_\nu^2\mathbf{E}_\nu$

the eigenmodes related to the 1st passband are some of these these modes, but there are more passbands and there are static modes

the expansion into eigenmodes is based on the completeness of the mode description

$$\mathbf{E}(r,z,t) = \sum \alpha_\nu(t)\mathbf{E}_\nu(r,z)$$

eigenmode ansatz in curl-curl equation

$$\mu\varepsilon\sum\left(\omega_\nu^2 + \frac{\partial^2}{\partial t^2}\right)\alpha_\nu(t)\mathbf{E}_\nu(\mathbf{r}) = -\frac{\partial}{\partial t}\mu\mathbf{J}$$

orthogonality of eigenmodes

$$\frac{1}{2}\int \varepsilon \mathbf{E}_\nu \mathbf{E}_\mu dV = W_\nu \delta_{\nu\mu} = L W'_\nu \delta_{\nu\mu}$$

with W_ν the energy of mode ν
and W'_ν the energy per length

ODE for coefficient functions

$$\left(\frac{d}{dt^2} + \omega_\nu^2\right)\alpha_\nu(t) = -\frac{d}{dt} g_\nu(t) \qquad \text{with} \qquad g_\nu(t) = \frac{1}{2}\int \mathbf{E}_\nu \mathbf{J} dV$$

in particular

$$g_{\nu,c}(t) = \frac{qv}{2} E_{\nu z}(r_s)\cos(k_\nu vt)\Phi(t)$$

$$\alpha_{\nu,c}(t) = -\frac{qv}{2} E_{\nu z}(r_s)\, f_c(t,\omega_\nu,k_\nu v)$$

$$f_c(t,a,b) = \frac{a\sin(at) - b\sin(bt)}{a^2 - b^2}$$

for PEC boundaries

$$g_{\nu,s}(t) = \frac{qv}{2} E_{\nu z}(0)\sin(k_\nu vt)\Phi(t)$$

$$a_{\nu,s}(t) = -\frac{qv}{2} E_{\nu z}(0)\, f_s(t,\omega_\nu,k_\nu v)$$

$$f_s(t,a,b) = -b\frac{\cos(at) - \cos(bt)}{a^2 - b^2}$$

for PMC boundaries

finally

$$E_{zc}\left(r_s,r_t,z,t\right)=\frac{q\beta}{2\pi\varepsilon_0}\sum A_\nu\left(r_s,r_t\right)\frac{-\dfrac{\omega_\nu}{c}\sin\left(\omega_\nu t\right)+k_\nu\beta\sin\left(k_\nu\beta ct\right)}{\left(\dfrac{\omega_\nu}{c}\right)^2-\left(k_\nu\beta\right)^2}\cos\left(k_\nu z\right)\Delta k$$

PEC

$$E_{zs}\left(r_s,r_t,z,t\right)=\frac{q\beta}{2\pi\varepsilon_0}\sum A_\nu\left(r_s,r_t\right)k_\nu\beta\frac{\cos\left(\omega_\nu t\right)-\cos\left(k_\nu\beta ct\right)}{\left(\dfrac{\omega_\nu}{c}\right)^2-\left(k_\nu\beta\right)^2}\sin\left(k_\nu z\right)\Delta k$$

PMC

with

$$\beta=v/c$$

$$\Delta k=\frac{\pi}{L}$$

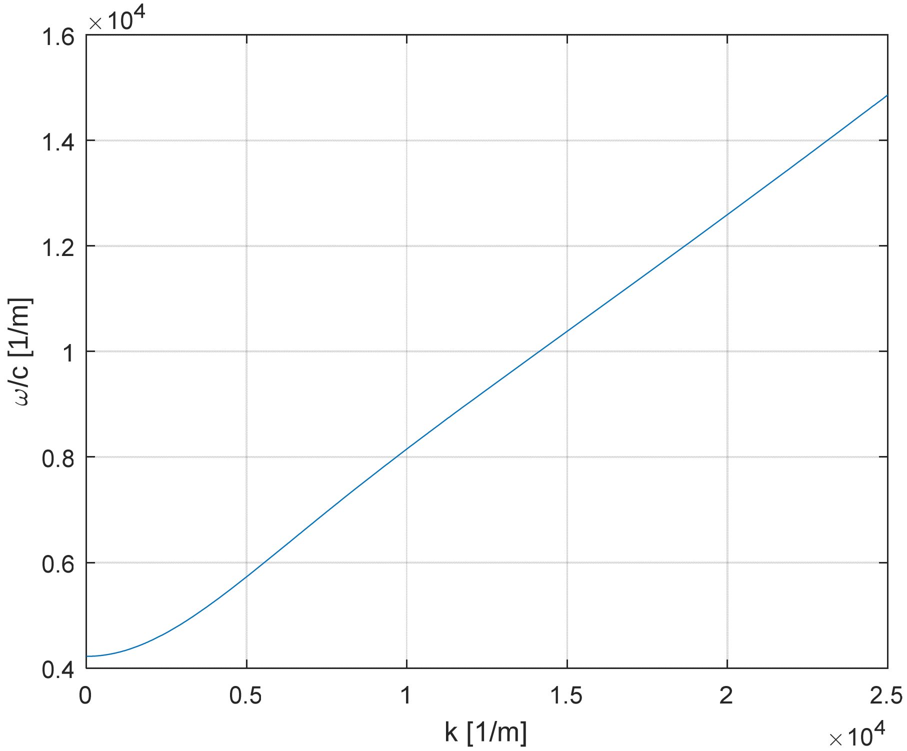
$$k_\nu=\nu\Delta k$$

$$A_\nu\left(r_s,r_t\right)=J_0\left(K_{0\nu}r_s\right)J_0\left(K_{0\nu}r_t\right)\frac{\left(E_{\nu z}\left(0\right)\right)^2}{W_\nu'}$$

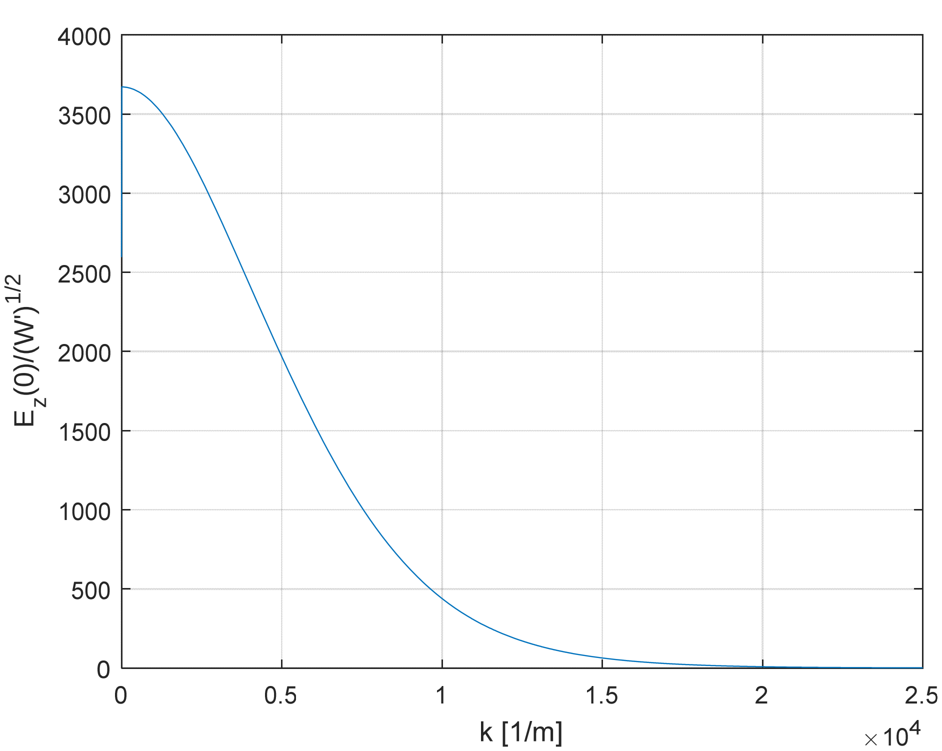
$$K_{0\nu}=\sqrt{\left(\frac{\omega}{c}\right)^2-k_\nu^2}$$

two functions are needed to use this result: $\omega_\nu=\omega\left(k_\nu\right)$ and $E_{\nu z}\left(0\right)/\sqrt{W_\nu'}$
(per pass-band)

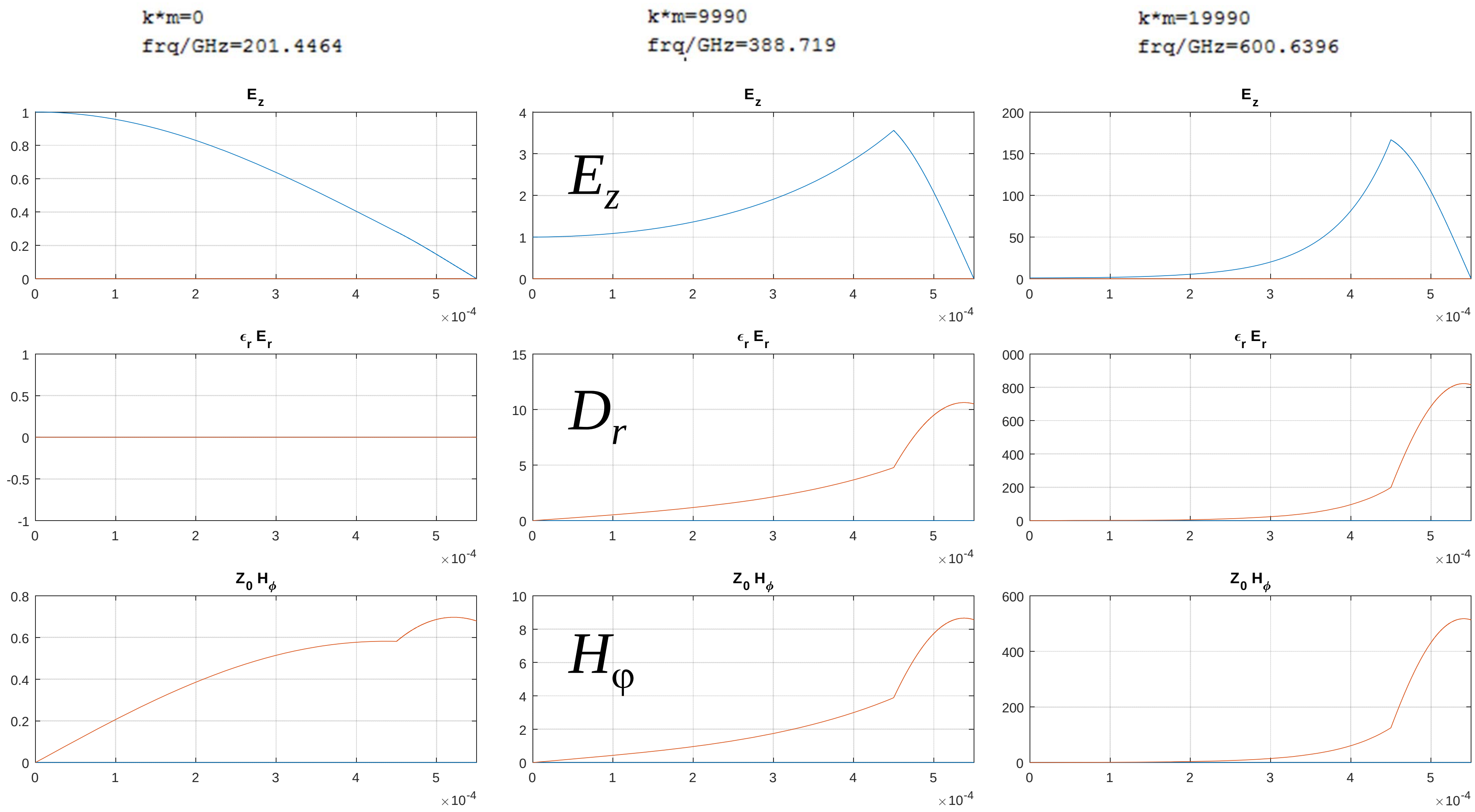
$$\omega_\nu = \omega(k_\nu)$$



$$E_{\nu z}(0)\Big/\sqrt{W'_\nu}$$



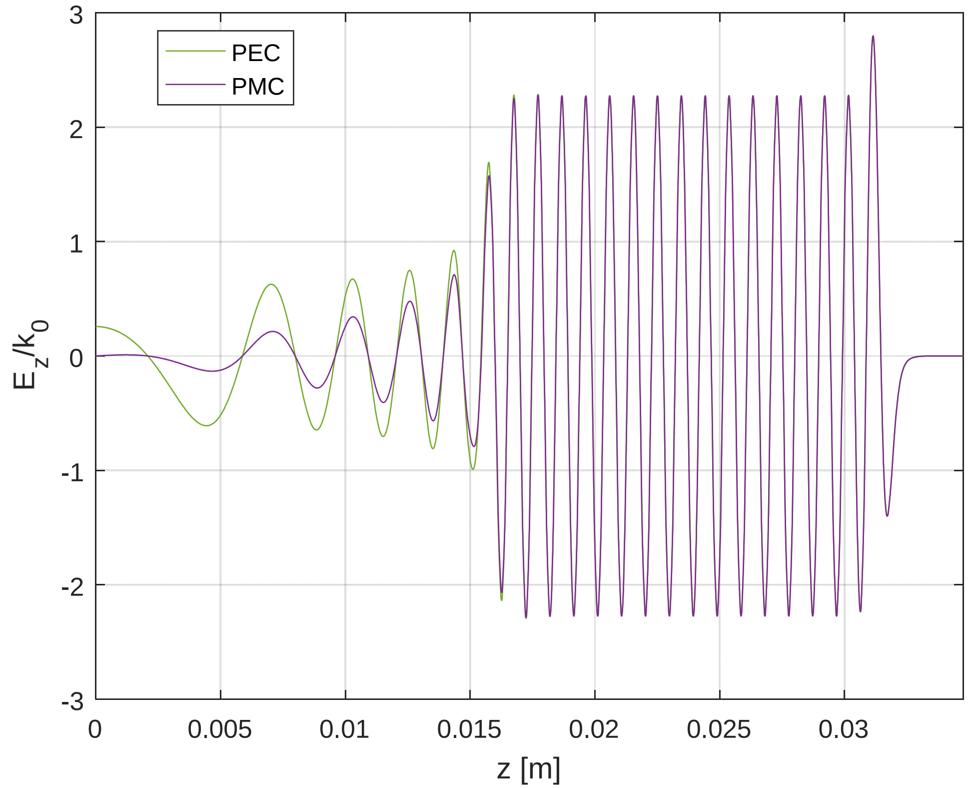
for 1st monopole band with r_1 = 0.45 mm, r_2 = 0.55 mm and ε_r = 3.8



for 1st monopole band with r_1 = 0.45 mm, r_2 = 0.55 mm and ε_r = 3.8

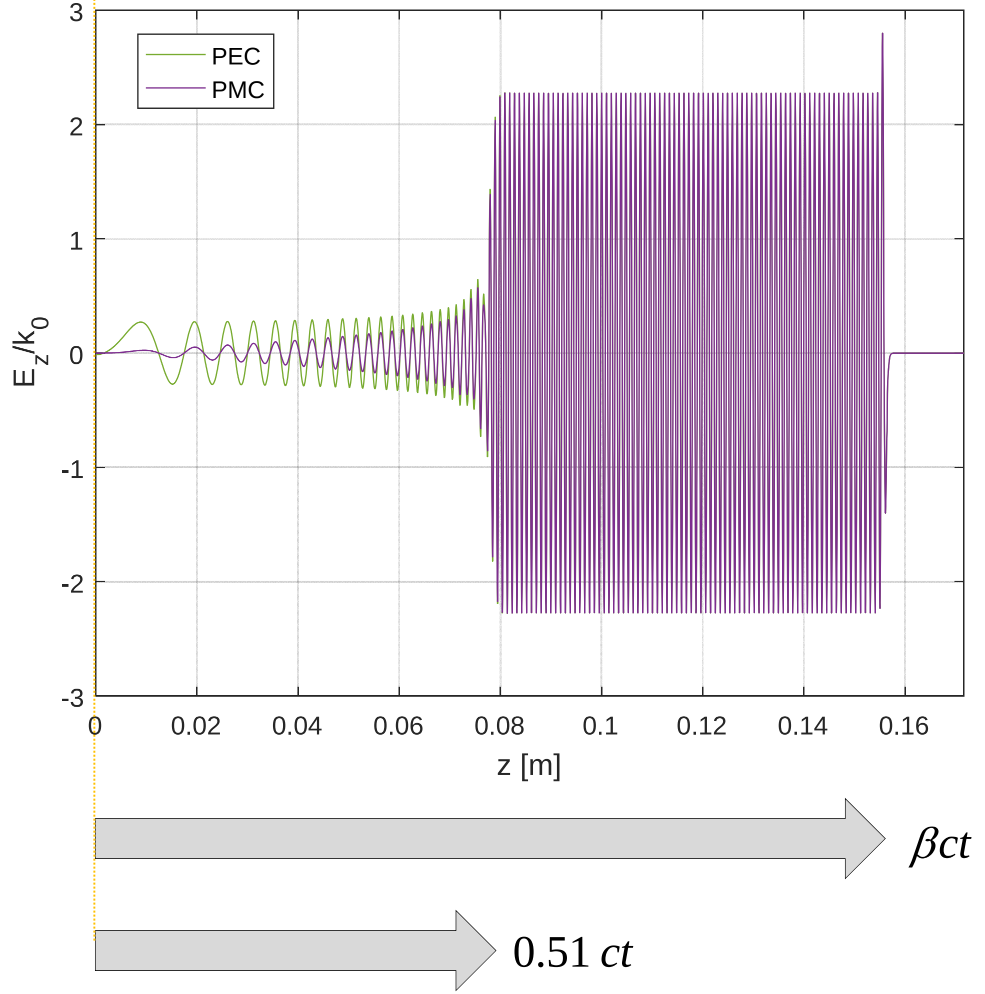
```
r1    = 0.00045
r2    = 0.00055
eps_r = 3.8
L     = 0.31416
c*t   = 0.031916
beta  = 0.99
```

≈ 30 λ

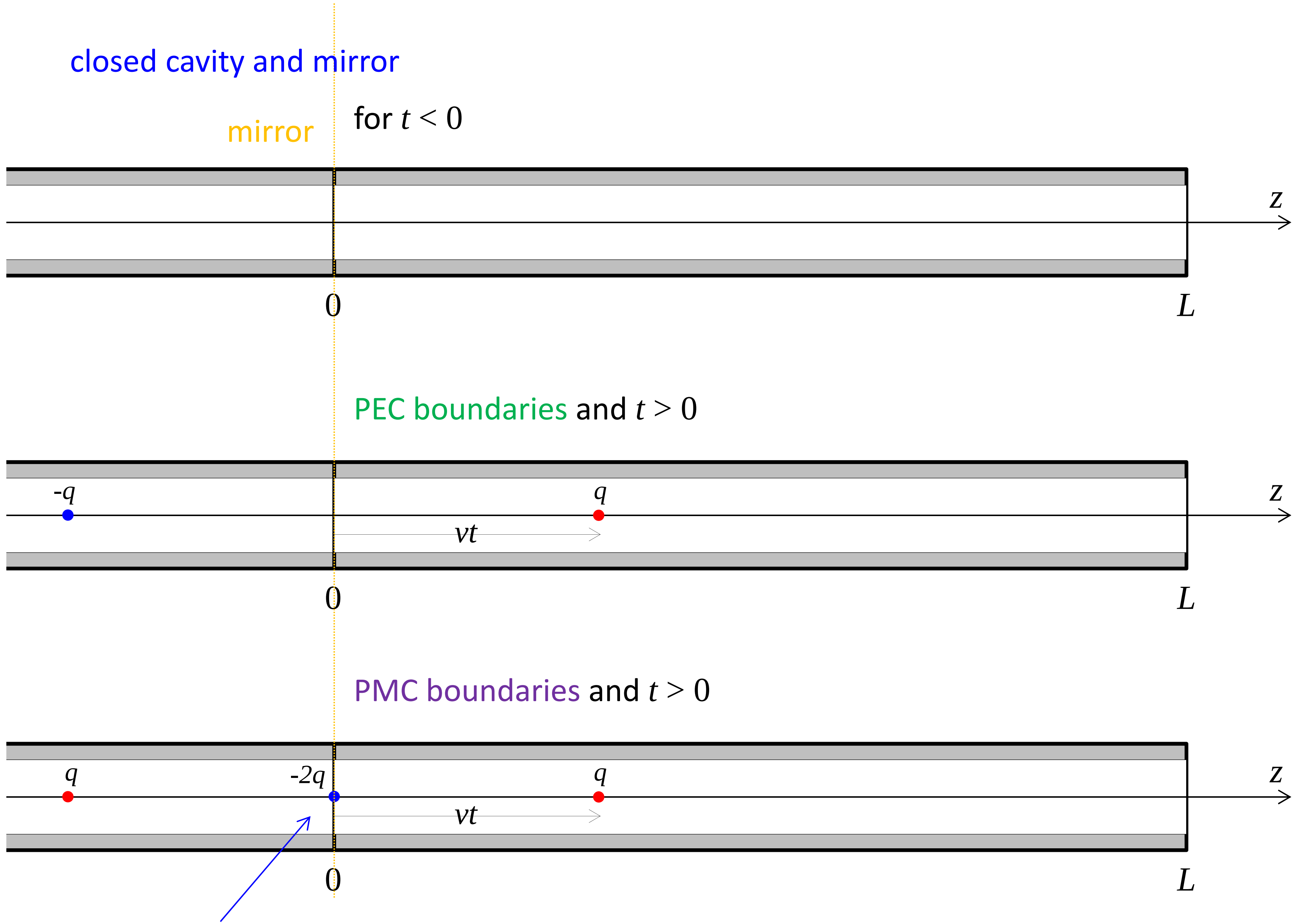


```
r1    = 0.00045
r2    = 0.00055
eps_r = 3.8
L     = 0.31416
c*t   = 0.15758
beta  = 0.99
```

≈ 150 λ



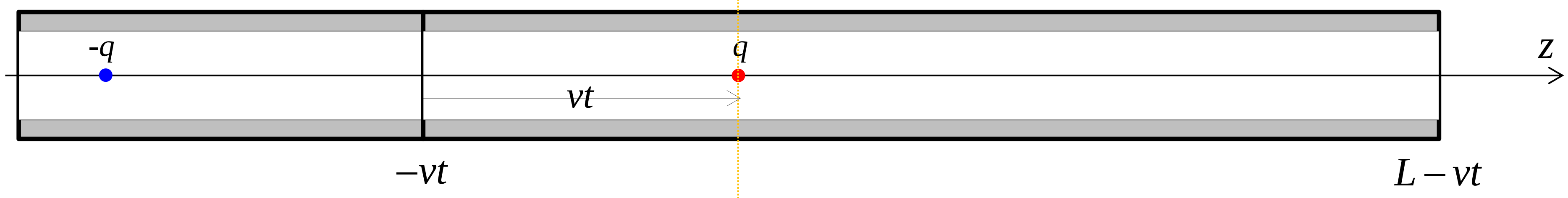
this is close to the group velocity (0.496c)



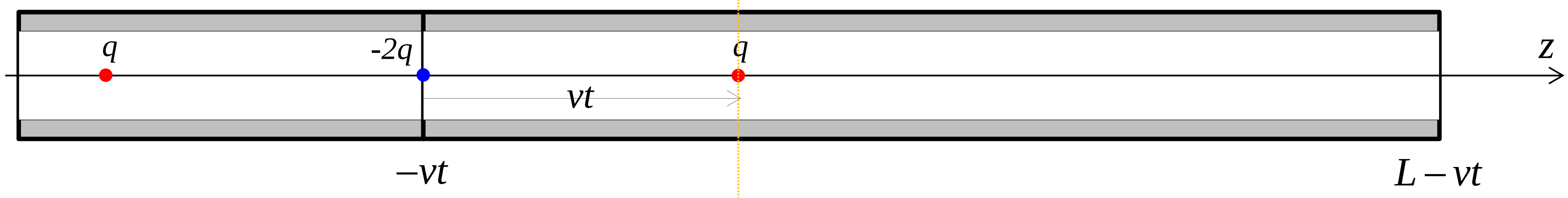
this charge is static; its field is curl-free; it is not seen by the fields of the 1st pass-band combining the results

new origin

PEC boundaries and $t > 0$



PMC boundaries and $t > 0$



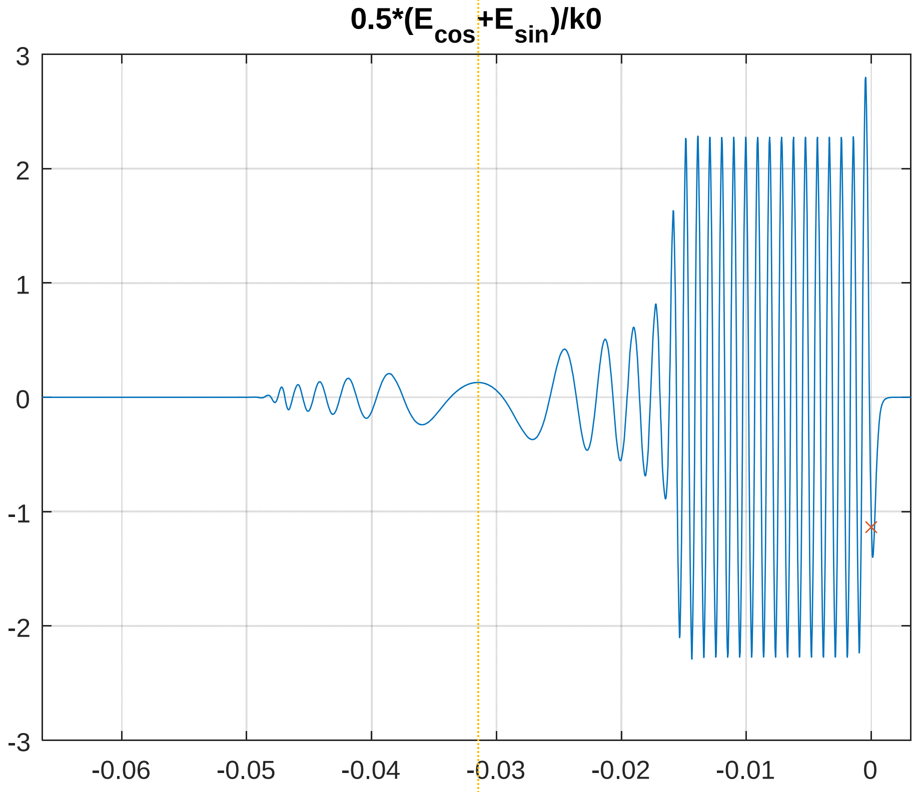
$$\mathbf{E} = \frac{\mathbf{E}_{PEC} + \mathbf{E}_{PMC}}{2}$$

the field (related to the 1st monopole band) shows the effect of a charge the was in rest for t <0 and that is in uniform motion for t>0;
it does not see the static charge in rest

```
r1     = 0.00045
r2     = 0.00055
eps_r  = 3.8
L      = 0.31416
c*t    = 0.031916
beta   = 0.99
```

≈ 30 λ

old origin



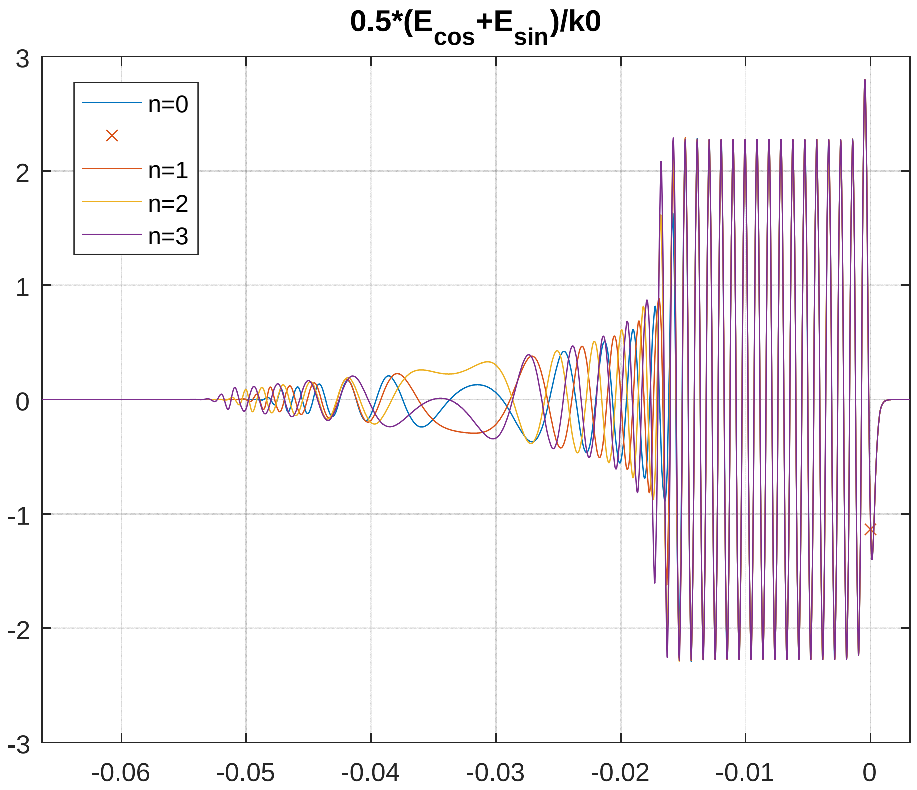
backward waves

standing waves

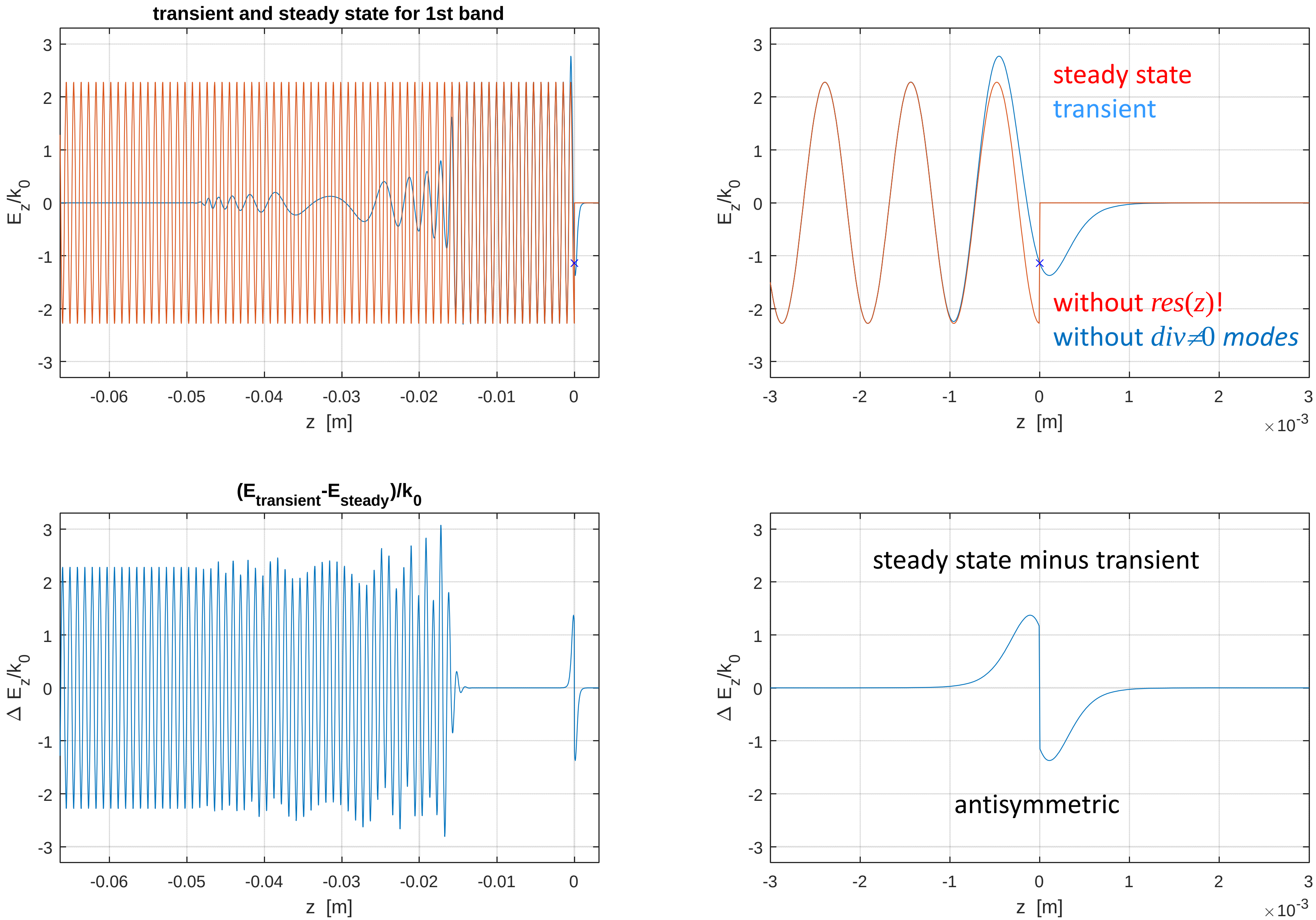
forward waves

```
r1     = 0.00045
r2     = 0.00055
eps_r  = 3.8
L      = 0.31416
c*t    = 0.031916 + n*lambda
beta   = 0.99
```

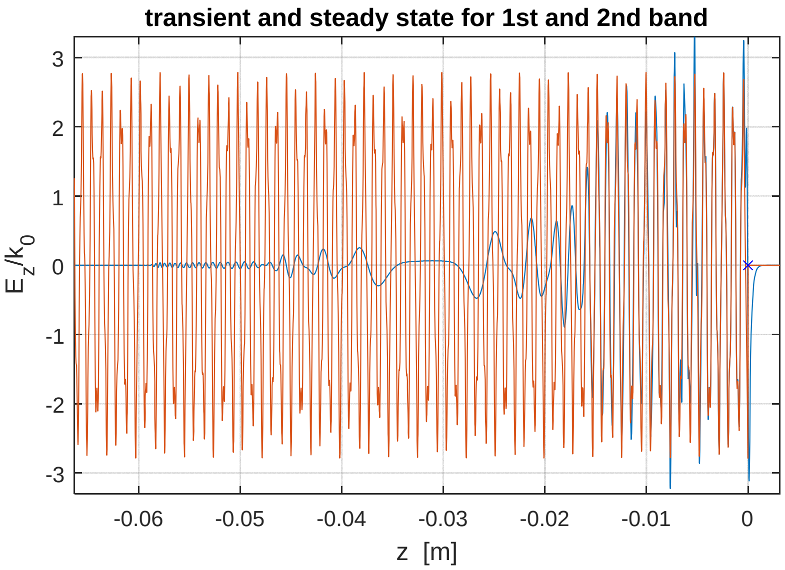
several snapshots

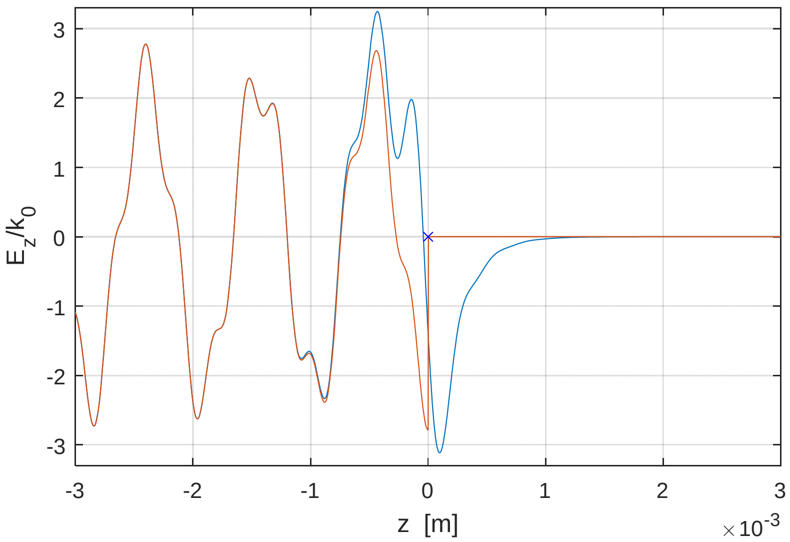


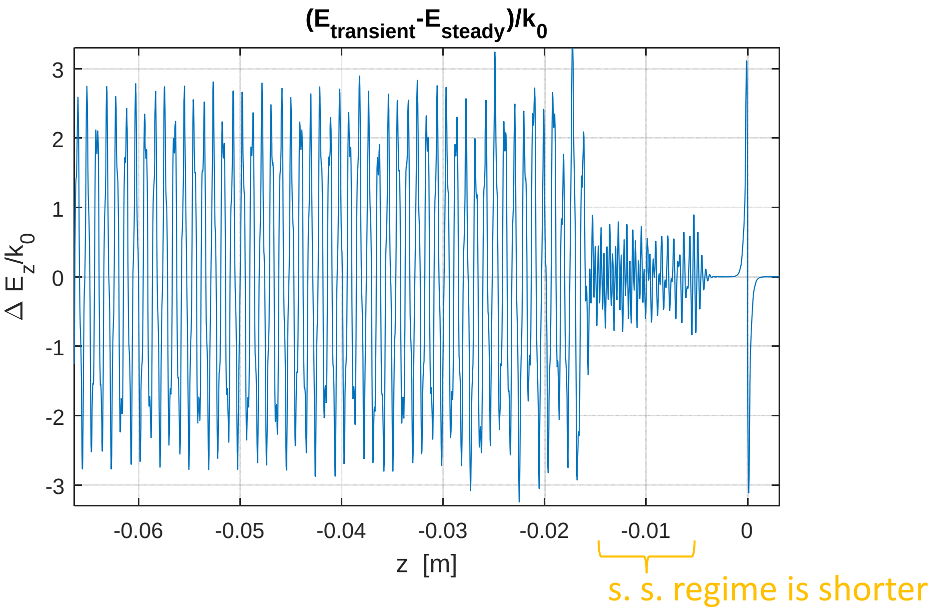
Comparison for the Steady State Regime

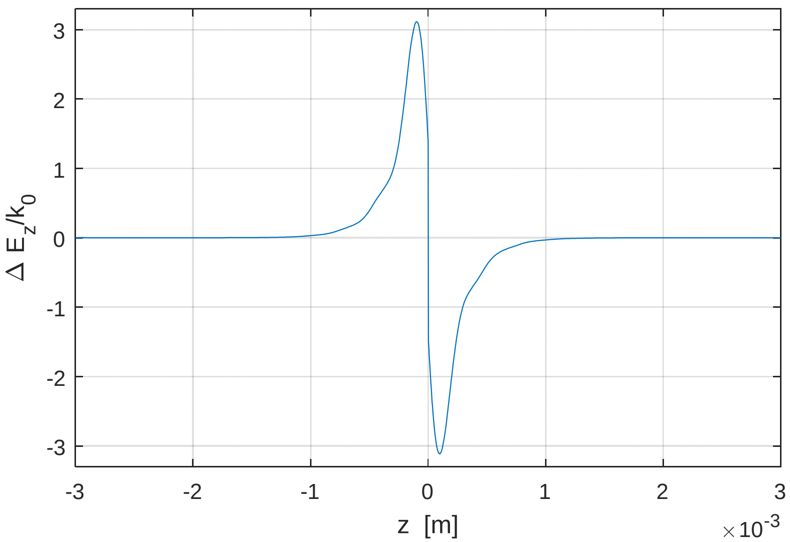


two bands

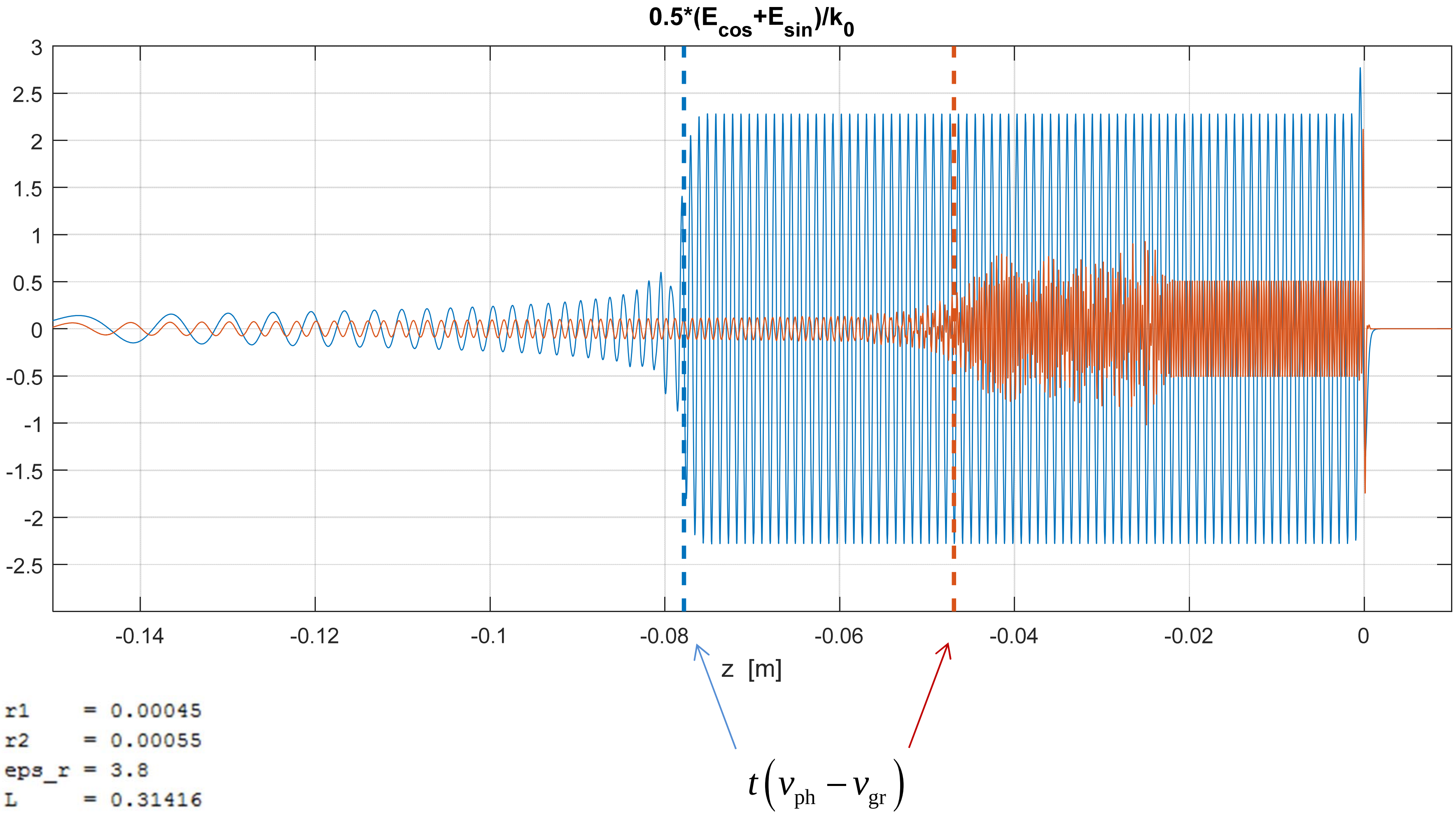

transient and steady state for 1st and 2nd band
E_z/k_0
z [m]


E_z/k_0
z [m]
×10^-3


(E_transient-E_steady)/k_0
Δ E_z/k_0
z [m]
s. s. regime is shorter


Δ E_z/k_0
z [m]
×10^-3

First and Second Passband in Transient Regime

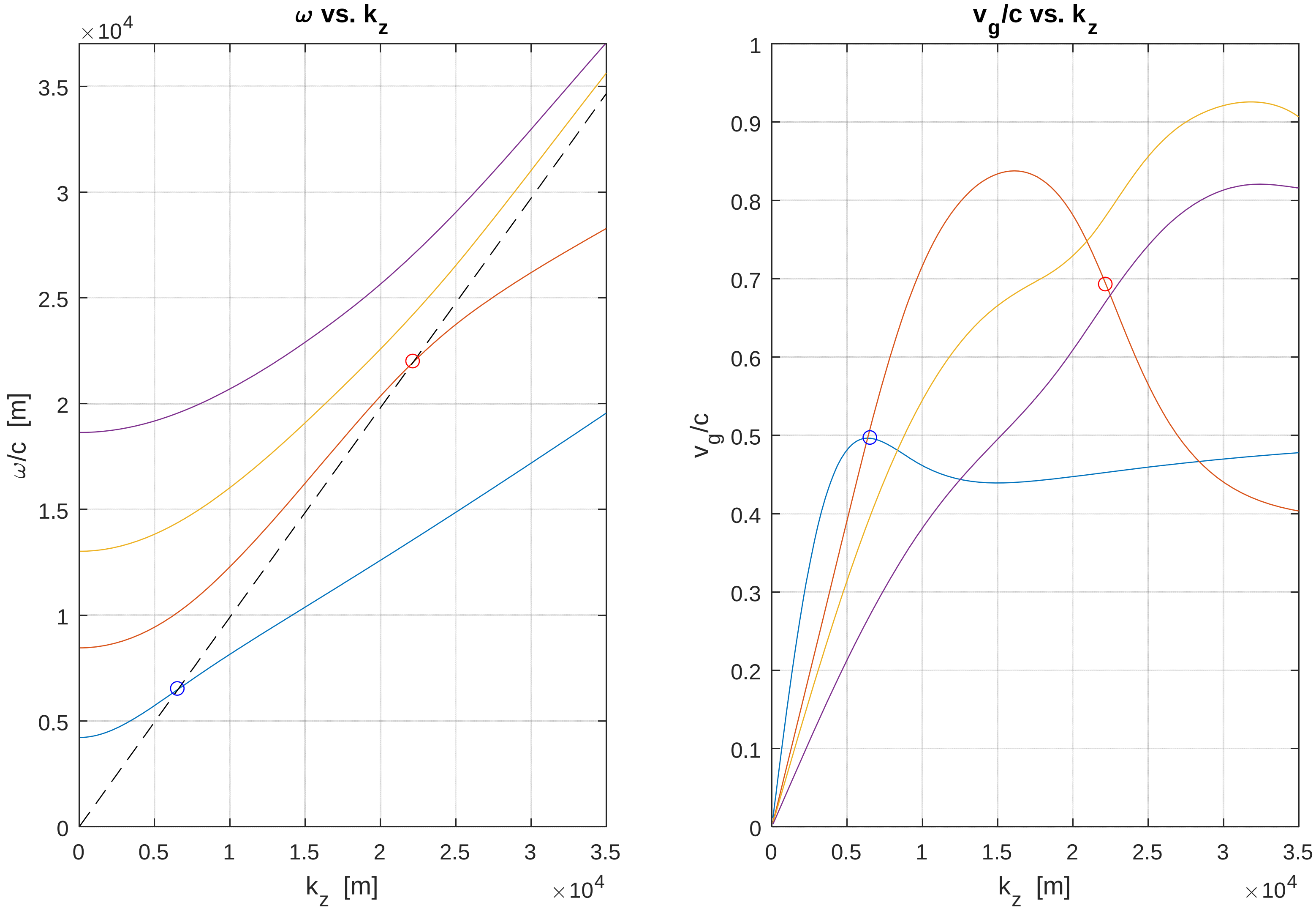



```
r1    = 0.00045
r2    = 0.00055
eps_r = 3.8
L     = 0.31416
c*t   = 0.15758
beta  = 0.99
```

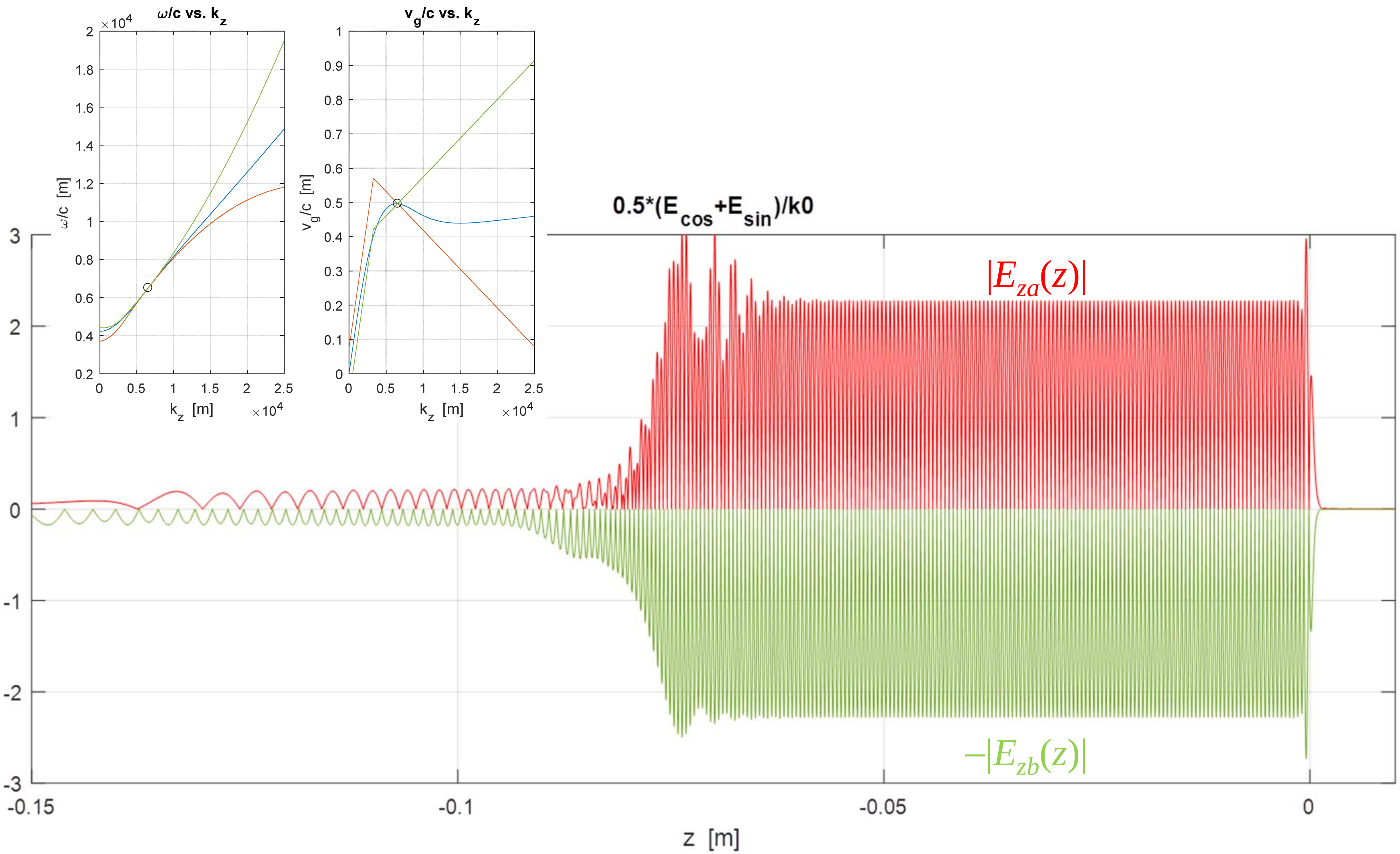
Why is the “burst” much nicer for the 1st band than for the 2nd?

Brillouin diagram and group velocity
(for the first 4 monopole bands)

```
r1     = 0.00045
r2     = 0.00055
eps_r = 3.8
```



calculation with manipulated dispersion diagram



yes, the 2nd derivative of the dispersion affects the shape of the burst

Summary and Conclusion

a loss free layered waveguide was investigated

steady state part by Fourier method

transient part by Cavity-Eigenmode-Method

good agreement of the resonant parts

missing parts are the resiudum of the pole expansion (Fourier method) and the contribution of divergent eigenmodes (CE-Method); both are easy to be calculated, but I was not interested in that

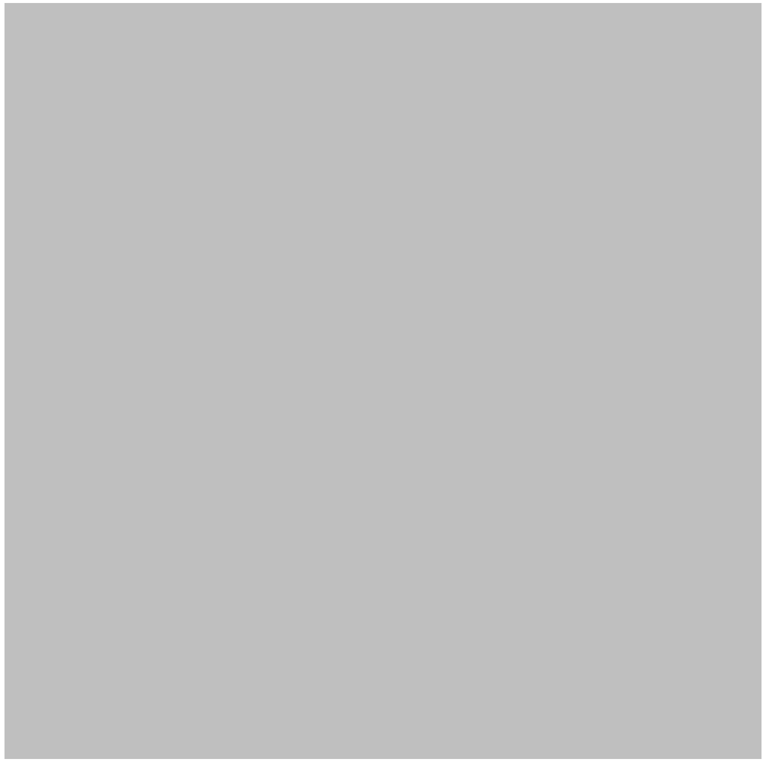
the shape of the burst is defined by the dispersion function $\omega(k)$ and the normalized on-axis amplitude $E_z(k)/\mathrm{sqrt}(W'(k))$

in particular the Taylor coefficients are important for the shape:

$$\omega(k) = \omega(k_r) + \omega'(k_r)(k - k_r) + \frac{1}{2}\omega''(k_r)(k - k_r)^2 + \cdots$$

~ group velocity

~ derivative of vg



More Gymnastics

$$E_{zc}(r_s,r_t,z,t)=\frac{q\beta}{2\pi\varepsilon_0}\sum A_\nu(r_s,r_t)\frac{-\frac{\omega_\nu}{c}\sin(\omega_\nu t)+k_\nu\beta\sin(k_\nu\beta ct)}{\left(\frac{\omega_\nu}{c}\right)^2-(k_\nu\beta)^2}\cos(k_\nu z)\Delta k$$

PEC

$$E_{zs}(r_s,r_t,z,t)=\frac{q\beta}{2\pi\varepsilon_0}\sum A_\nu(r_s,r_t)k_\nu\beta\frac{\cos(\omega_\nu t)-\cos(k_\nu\beta ct)}{\left(\frac{\omega_\nu}{c}\right)^2-(k_\nu\beta)^2}\sin(k_\nu z)\Delta k$$

PMC

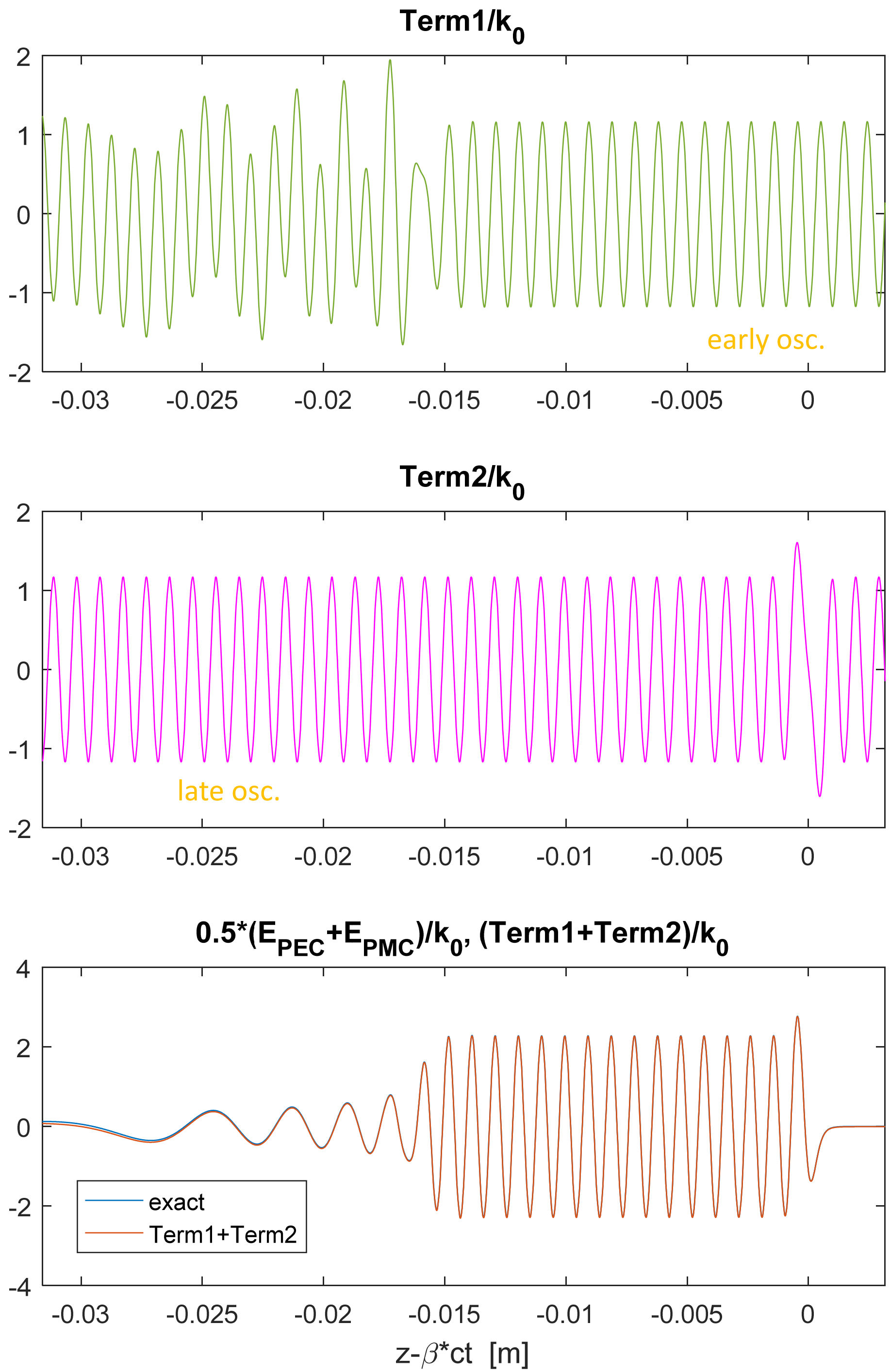
for $\beta ct \approx z >> \lambda$

neglect terms with fast oscillation in z, with $s = \beta ct - z$

$$\frac{E_{zs}+E_{zc}}{2}\rightarrow -\frac{q\beta}{4\pi\varepsilon_0}\int_0^\infty A(k)\frac{\sin\left(\left(\frac{\omega(k)}{\beta c}-k\right)z+\frac{\omega(k)}{\beta c}s\right)}{\frac{\omega(k)}{c}-k\beta}dk+\frac{q\beta^2}{2\pi\varepsilon_0}\int_0^\infty A(k)\frac{k\sin(ks)}{\left(\frac{\omega(k)}{c}\right)^2-(k\beta)^2}dk$$

term 1, late transients

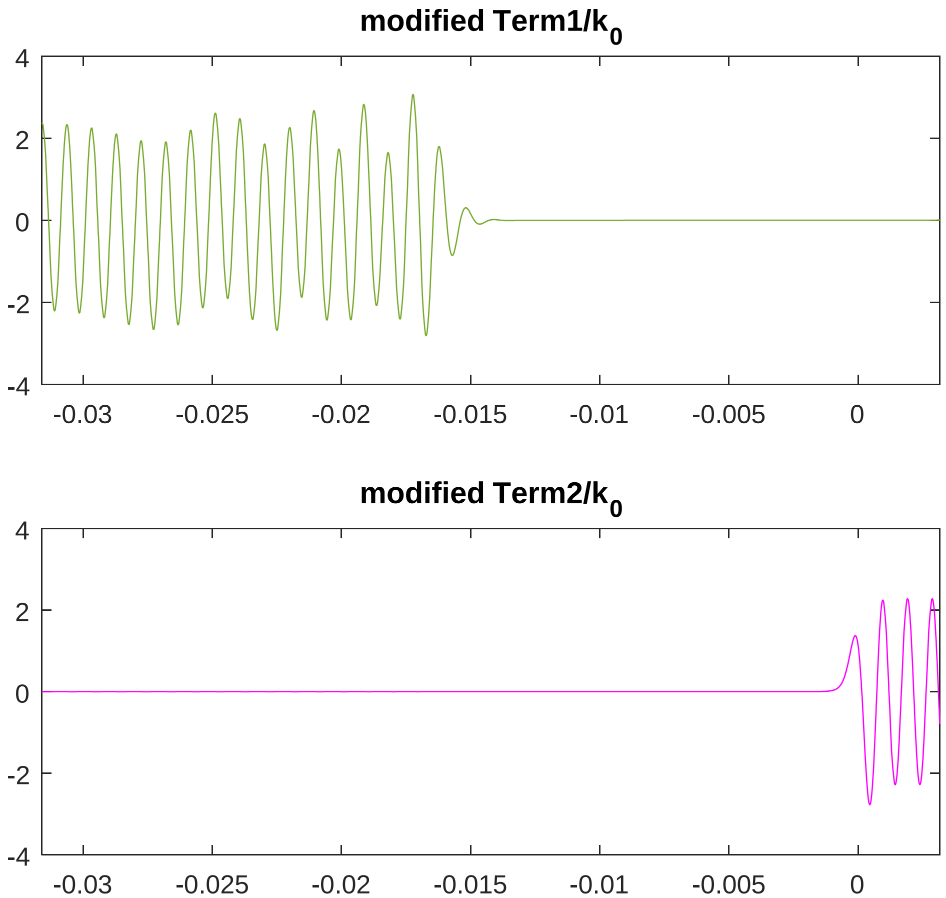
term 2, early transients



more gymnastic exercises:

extract singularities from the integrals → modified Term1 and Term2 without early/late oscillation

for $ct \approx 30\ \lambda$


modified Term1/k_0
4
2
0
-2
-4
-0.03
-0.025
-0.02
-0.015
-0.01
-0.005
0
modified Term2/k_0
4
2
0
-2
-4
-0.03
-0.025
-0.02
-0.015
-0.01
-0.005
0