E 106

Hohlraumresonatoren / Cavities

Details on the experimental method

E 106 Hohlraumresonatoren (Cavities)

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1. Cavities

We derive the natural oscillations (i.e. resonant modes) of a waveguide from the Maxwell-equations. First considering the wave propagation in cylindrical waveguides, we introduce the different modes for the propagation of waves (i.e. modes of a wave guide). The transition to a cavity is made by closing the waveguide with two conducting plates. This introduces additional (longitudinal) boundary conditions and causes the formation of plane waves.

1.1. Propagation of Waves and Maxwell-equations

We start with the Maxwell-equations in their differential form, which in vacuum (no charges or currents present) can be expressed in terms of the E- and H-Fields using ε_0 and μ_0.

$$\boxed{\begin{array}{lcl}
rot\,\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \overset{\text{vacuum}}{\mapsto} & \vec{\nabla}\times\vec{E} = -\mu_0\cdot\dfrac{\partial \vec{H}}{\partial t} \\
div\,\vec{D} = \rho & \overset{\text{vacuum}}{\mapsto} & \vec{\nabla}\bullet\vec{E} = 0 \\
rot\,\vec{H} = \vec{j} + \dfrac{\partial \vec{D}}{\partial t} & \overset{\text{vacuum}}{\mapsto} & \vec{\nabla}\times\vec{H} = \varepsilon_0\cdot\dfrac{\partial \vec{E}}{\partial t} \\
div\,\vec{B} = 0 & \overset{\text{vacuum}}{\mapsto} & \vec{\nabla}\bullet\vec{H} = 0
\end{array}}$$

By taking the curl of the first (third) equation, replacing $\vec{\nabla}\times\dfrac{\partial \vec{H}}{\partial t}$ $(\vec{\nabla}\times\dfrac{\partial \vec{E}}{\partial t})$ by the time derivative of the third (first) equation and using the identity

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{a}\right) = \vec{\nabla}\left(\vec{\nabla}\bullet\vec{a}\right) - \Delta\vec{a}\,,$$

as well as the divergence relations (eqs. 2 and 4) we get the differential equations of the electric and magnetic field for the propagation of waves in vacuum:

$$\begin{aligned}
\Delta\vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} &= 0 \\
\Delta\vec{H}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{H}(\vec{r},t)}{\partial t^2} &= 0
\end{aligned}
\qquad \text{mit} \quad c^2 = \frac{1}{\mu_0\,\varepsilon_0}$$

If we consider only waves with a fixed frequency ω, we can express the time-dependence explicitly by

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\cdot e^{i\omega t}, \quad \vec{H}(\vec{r},t) = \vec{H}(\vec{r})\cdot e^{i\omega t},$$

and simplify the wave equation by inserting this solution:

$$\Delta\vec{E}(\vec{r}) + \frac{\omega^2}{c^2}\vec{E}(\vec{r}) = 0$$
$$\Delta\vec{H}(\vec{r}) + \frac{\omega^2}{c^2}\vec{H}(\vec{r}) = 0$$

1.2. Waveguides

We begin with a general waveguide, aligned in z-direction. This means that the propagation of waves is also fixed to the z-direction. The ansatz

$\vec{E} = \vec{E}(x,y)\cdot e^{i(\omega t - kz)}$ and the separation $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$ yields for the longitudinal fields:

$$\begin{aligned}\Delta_\perp E_z + k_c{}^2 E_z &= 0\\ \Delta_\perp H_z + k_c{}^2 H_z &= 0\end{aligned} \qquad \text{mit} \qquad \boxed{\frac{\omega^2}{c^2} - k^2 = k_c{}^2}$$

(dispersion relation of the waveguide)

The quantity k_c is called critical wave number and is a characteristic of the cavity, as we shall see in the following.

For the calculation of the transversal fields we use the first (third) Maxwell-Equation. **It will turn out that it is sufficient to know the longitudinal fields, E_z and H_z, since the corresponding transversal fields, $\vec{E}_\perp$ und $\vec{H}_\perp$, can be calculated using the longitudinal ones**.

For the E-field for instance, we get:

$$\left.\begin{aligned}\left(\vec{\nabla}\times\vec{E}_\perp\right)_x &= -\frac{\partial E_y}{\partial z} = +ikE_y\\ \left(\vec{\nabla}\times\vec{E}_\perp\right)_y &= +\frac{\partial E_x}{\partial z} = -ikE_x\end{aligned}\right. \Rightarrow \left(\vec{\nabla}\times\vec{E}_\perp\right)_\perp = ik\vec{E}_\perp\times\hat{e}_z$$

$$\vec{\nabla}\times\left(E_z\cdot\hat{e}_z\right) = \vec{\nabla}E_z\times\hat{e}_z = \vec{\nabla}_\perp E_z\times\hat{e}_z = \vec{\nabla}_\perp\times\left(E_z\cdot\hat{e}_z\right)$$

The combination of these two equations yields

$$\left(\vec{\nabla}\times\vec{E}\right)_\perp = \left(ik\vec{E}_\perp + \vec{\nabla}_\perp E_z\right)\times\hat{e}_z$$

and in complete analogy $\left(\vec{\nabla}\times\vec{H}\right)_\perp = \left(ik\vec{H}_\perp + \vec{\nabla}_\perp H_z\right)\times\hat{e}_z$.

From the first and third Maxwell-Equations we get

$$\begin{aligned}\left(ik\vec{E}_\perp + \vec{\nabla}_\perp E_z\right)\times\hat{e}_z &= -i\omega\mu_0\vec{H}_\perp \\ \left(ik\vec{H}_\perp + \vec{\nabla}_\perp H_z\right)\times\hat{e}_z &= i\omega\varepsilon_0\vec{E}_\perp\end{aligned} \quad \text{oder} \quad \begin{aligned}ik\vec{E}_\perp + \vec{\nabla}_\perp E_z &= i\omega\mu_0\vec{H}_\perp\times\hat{e}_z \\ ik\vec{H}_\perp + \vec{\nabla}_\perp H_z &= -i\omega\varepsilon_0\vec{E}_\perp\times\hat{e}_z\end{aligned}$$

By some simple calculations (crosswise substitution) this can be converted into the following relations:

$$\boxed{\begin{aligned}ik_c^2\vec{E}_\perp &= k\vec{\nabla}_\perp E_z + \omega\mu_0\vec{\nabla}_\perp H_z\times\hat{e}_z \\ ik_c^2\vec{H}_\perp &= k\vec{\nabla}_\perp H_z - \omega\varepsilon_0\vec{\nabla}_\perp E_z\times\hat{e}_z\end{aligned}}$$

We can classify the different possible waves as follows:

a) $k_c^2 = 0$:

Phase velocity from the dispersion relation: $\boxed{\mathrm{v}_{ph} = \frac{\omega}{k} = c}$

Impedance via $\zeta = \frac{E}{H}$, then $\vec{\nabla}_\perp E_z = -\zeta\cdot\vec{\nabla}_\perp H_z\times\hat{e}_z$: $\boxed{\zeta = \sqrt{\frac{\mu_0}{\varepsilon_0}}}$

1.) $\vec{\nabla}_\perp E_z \neq 0,\ \vec{\nabla}_\perp H_z \neq 0$: HE or EH hybrid waves (are used for deflection of charged particles in HF separators)

2.) $\vec{\nabla}_\perp E_z = 0$ and $\vec{\nabla}_\perp H_z = 0$: **transversal TEM waves**

b) $k_c^2 \neq 0$:

No propagation for $\omega \leq c\cdot k_c$: evanescent waves ↔ "cut-off"

Phase velocity from the dispersion relation: $\boxed{\mathrm{v}_{ph} = c\cdot\sqrt{1+\frac{k_c^2}{k^2}} > c}$

The Impedance depends on the propagation mode:

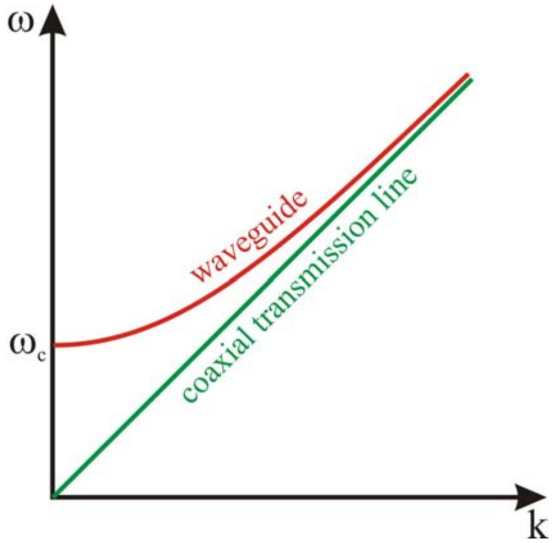
1.) $E_z = 0$: TE (transversal electric) or H (because $H_z \neq 0$) waves

Impedance via $ik\vec{E}_\perp = -i\omega\mu_0\vec{H}_\perp \times \hat{e}_z$: $\boxed{Z_0 = \frac{E_\perp}{H_\perp} = \mu_0 \frac{\omega}{k}}$

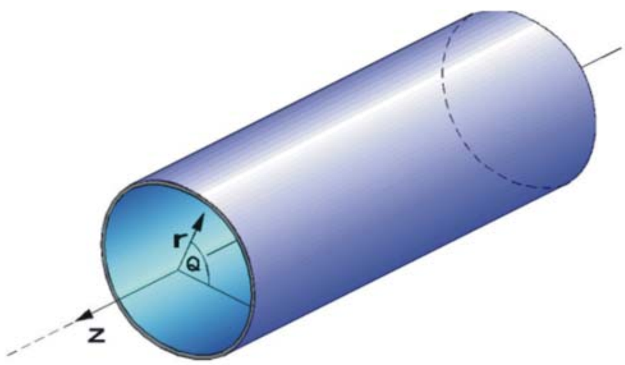
2.) $H_z = 0$: TM (transversal magnetic) or E (because $E_z \neq 0$) waves

Impedance via $ik\vec{H}_\perp \times \hat{e}_z = \left(i\omega\varepsilon_0 + \sigma\right)\vec{E}_\perp$: $\boxed{Z_0 = \frac{k}{\omega\varepsilon_0}}$

Corresponding to the critical wave number there is a critical frequency, $\omega_c = k_c \cdot c$, below which there is no propagation of waves in the waveguide. For small frequencies the dispersion in the waveguide therefore differs clearly from the dispersion in vacuum or in a coaxial cable (TEM-waves):



In the following we shall consider a cylindrical waveguide with (inner) radius *a*:



The longitudinal fields have to fulfil the wave-equation:

$$\underbrace{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)}_{=\Delta_\perp}E_z+k_c{}^2E_z = 0$$

$$\overbrace{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)}H_z+k_c{}^2H_z = 0$$

Separation of the φ-dependence,

$$E_z(r,\varphi) = R_E(r)\cdot\theta_E(\varphi),\quad H_z(r,\varphi) = R_H(r)\cdot\theta_H(\varphi)$$

$$\Rightarrow \qquad \frac{r^2}{R}\cdot\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+k_c{}^2r^2 = -\frac{1}{\theta}\frac{d^2\theta}{d\varphi^2} = m^2,$$

yields two differential equations, one for the angular and one for the radial dependence of the longitudinal component. They are solved by the trigonometric functions and the Bessel/Neumann functions, respectively.

$$\begin{aligned}&\frac{d^2\theta}{d\varphi^2}+m^2\theta = 0 \\ &\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(k_c{}^2-\frac{m^2}{r^2}\right)R = 0\end{aligned} \quad\rightarrow\quad \begin{aligned}&\theta(\varphi) = A\cdot\cos(m\varphi)+B\cdot\sin(m\varphi) \\ &R(r)=C\cdot J_m(k_c\cdot r)+D\cdot N_m(k_c\cdot r)\end{aligned}$$

With $\vec{\nabla}_\perp=\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}$ we get for the transversal fields:

$$\boxed{\begin{aligned} ik_c{}^2\vec{E}_\perp &= k\left\{\frac{\partial E_z}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial E_z}{\partial\varphi}\hat{e}_\varphi\right\}-\omega\mu_0\left\{\frac{\partial H_z}{\partial r}\hat{e}_\varphi-\frac{1}{r}\frac{\partial H_z}{\partial\varphi}\hat{e}_r\right\} \\ ik_c{}^2\vec{H}_\perp &= k\left\{\frac{\partial H_z}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial H_z}{\partial\varphi}\hat{e}_\varphi\right\}+\omega\varepsilon_0\left\{\frac{\partial E_z}{\partial r}\hat{e}_\varphi-\frac{1}{r}\frac{\partial E_z}{\partial\varphi}\hat{e}_r\right\}\end{aligned}}$$

The possible field distributions are further constrained by the boundary conditions at the walls of the waveguide:

- $E_\varphi=0;\ E_z=0$ für $r=a$ (vanishing tangential component)
- $H_r = 0$ für $r=a$ (vanishing normal component)

This suggests the following ansatz for the magnetic field and the electric field, in case of TE- and TM-modes, respectively:

TE- or H-waves with $E_z = 0$:

$H_z = H_{mn} \cdot J_m(k_c \cdot r) \cdot \cos(m\varphi)$, where $J'_m(k_c a) = 0$ has to be fulfilled und n specifies, which zero point j'_{mn} it is. With $n = 1,2,3,\ldots$ and $m = 0,1,2,\ldots$ there is the following approximate **dispersion relation for TE-Waves**:

$$k_c^2 + k^2 = \frac{\omega^2}{c^2} \qquad \text{where} \qquad (k_c a)^2 \approx \left(n + \frac{2m-3}{4}\right)^2 \pi^2 - \frac{4m^2+3}{4}$$

With this we get for the transversal fields of the TE-Waves:

$$E_r = i\frac{\omega\mu_0}{k_c}\frac{m}{k_c r} \cdot J_m(k_c r) \cdot \sin(m\varphi) H_{mn} \qquad H_r = -i\frac{k}{k_c} \cdot J'_m(k_c r) \cdot \cos(m\varphi) H_{mn}$$

$$E_\varphi = i\frac{\omega\mu_0}{k_c} \cdot J'_m(k_c r) \cos(m\varphi) H_{mn} \qquad H_\varphi = i\frac{k}{k_c}\frac{m}{k_c r} \cdot J_m(k_c r) \cdot \sin(m\varphi) H_{mn}$$

The lowest frequency mode (fundamental mode) is TE_{11} with m>0, due to the requirement that the normal component of H must vanish! In general, for the cut-off-frequencies we have: $\omega_{mn} = j_{mn} \cdot c/a$

TM- or E-Waves with $H_z = 0$:

$E_z = E_{mn} \cdot J_m(k_c \cdot r) \cdot \cos(m\varphi)$, where $J_m(k_c a) = 0$ has to be fulfilled and n specifies which the zero point j_{mn} it is. With $n = 1,2,3,\ldots$ and $m = 0,1,2,\ldots$ there is the following approximate **dispersion relation for TM-Waves**:

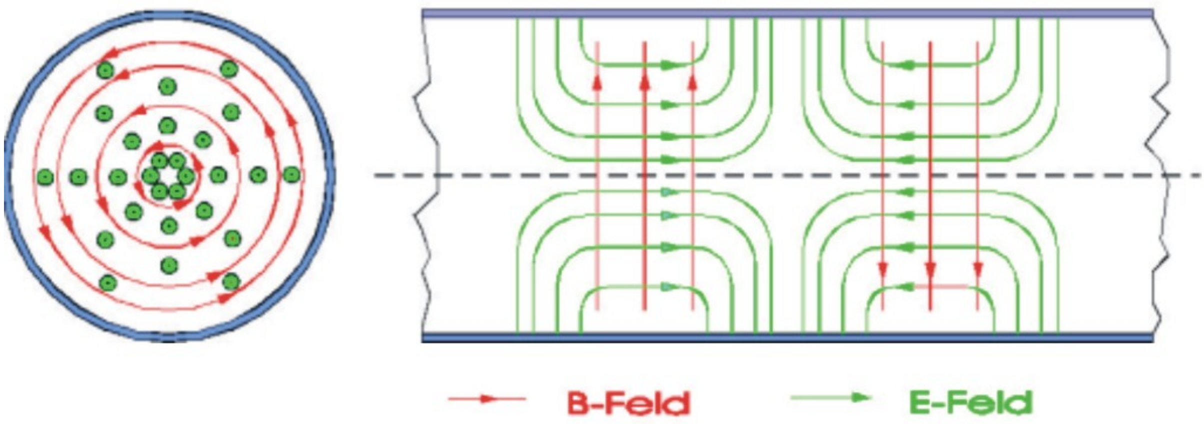
$$k_c^2 + k^2 = \frac{\omega^2}{c^2} \qquad \text{with} \qquad (k_c a)^2 \approx \left(n + \frac{2m-1}{4}\right)^2 \pi^2 - \frac{4m^2-1}{4}$$

With this we get for the transversal fields of the TM-Waves:

$$E_r = -i\frac{k}{k_c} \cdot J'_m(k_c r) \cdot \cos(m\varphi) E_{mn} \qquad H_r = i\frac{\omega\varepsilon_0}{k_c}\frac{m}{k_c r} \cdot J_m(k_c r) \cdot \sin(m\varphi) E_{mn}$$

$$E_\varphi = i\frac{k}{k_c}\frac{m}{k_c r} \cdot J_m(k_c r) \cdot \sin(m\varphi) E_{mn} \qquad H_\varphi = i\frac{\omega\varepsilon_0}{k_c} \cdot J'_m(k_c r) \cos(m\varphi) E_{mn}$$

The fundamental mode is TM_{01} with *m=0* and *n=1*, $E_\varphi = H_r = 0$:



The mode TM_{01} is used in linear accelerators and accelerating resonators for accelerating ultra-relativistic particles.

For the cut-off-frequencies we have: $\omega_{mn} = j_{mn} \cdot c/a$

1.3. Eigenmodes of cylindrical resonators

If we insert conducting plates into the waveguide perpendicular to the z-direction, the incoming wave is reflected completely and we get a standing wave. This changes the z-dependence of the fields:

$$a \cdot e^{ikz} \rightarrow A \cdot \sin(kz + \varphi_0)$$

At nodal planes, conducting plates can be inserted without changing the distribution of the fields. This leads to a cylindrical cavity, consisting of a waveguide of length *l* which is closed on both sides by conducting plates. To fulfil the longitudinal boundary conditions we have to impose the condition that $k = p \cdot \pi/l$.

Inserting the eigenmodes of the waveguide, the longitudinal fields become:

TE_{mnp}-Modes: $H_z = H_{mn} \cdot J_m(k_c r) \cdot \cos(m\varphi) \cdot \sin(p\pi/l \cdot z) \cdot e^{i\omega_{mnp}t}$

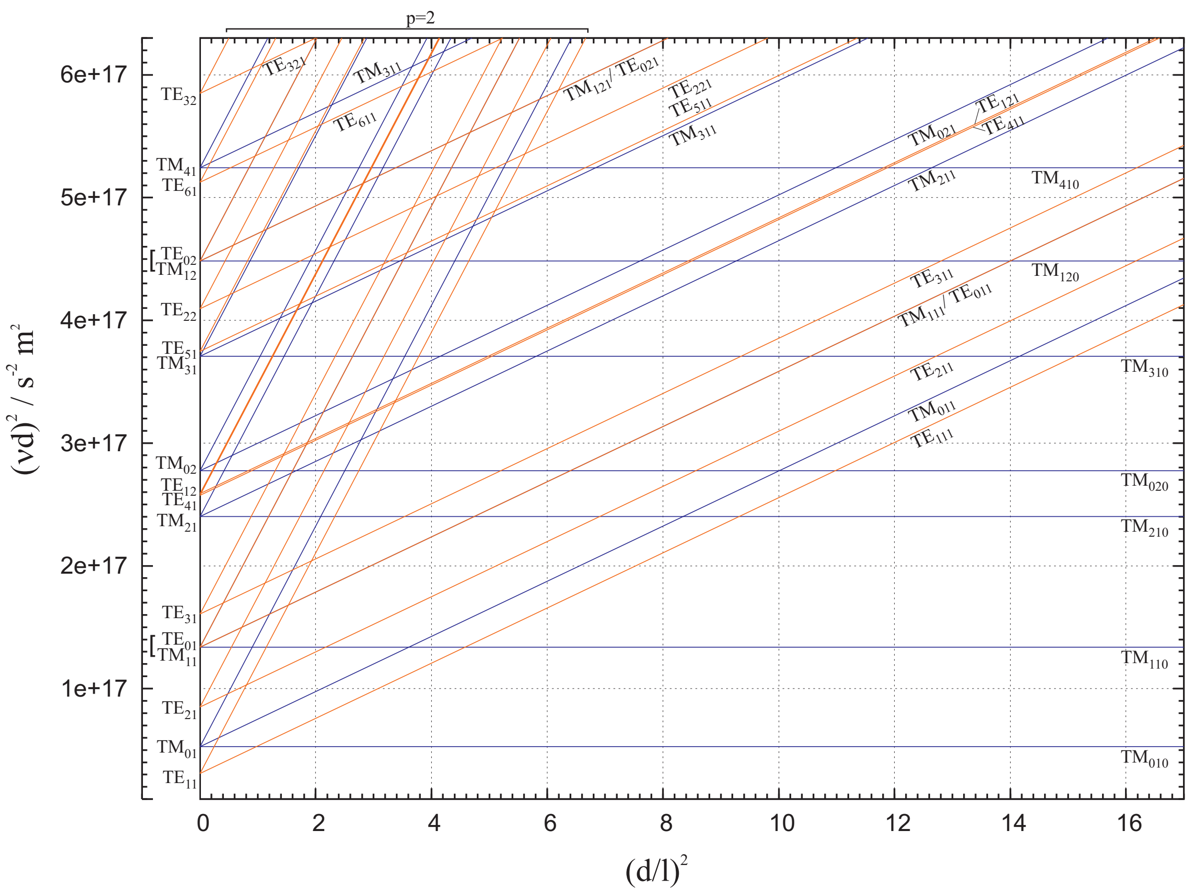
TM_{mnp}-Modes: $E_z = E_{mn} \cdot J_m(k_c r) \cdot \cos(m\varphi) \cdot \cos(p\pi/l \cdot z) \cdot e^{i\omega_{mnp}t}$

For the resonant frequencies one has: $\omega_{mnp} = c \cdot \sqrt{(j_{mn}/a)^2 + (p\pi/l)^2}$

The formula for the resonant frequencies can be written as a linear equation as follows:

$$(d\nu)^2 = \left(\frac{c j_{mn}^{(')}}{\pi}\right)^2 + \left(\frac{c}{2}\right)^2 p^2 \left(\frac{d}{l}\right)^2$$

Here $d = 2 \cdot a$ is the diameter of the cavity and $j_{mn}^{(')}$ denotes the zero point of the Bessel function or its derivative. Plotting the lines of the different modes in a diagram one gets the so called **mode map** (here for $p \leq 2$):



From this map, the structure of the different modes of a resonator for a given relation between diameter and length can be easily read off. The frequency can be determined from the ordinate.

The zero points of the Bessel functions and its first derivative which are necessary for the explicit calculation of the resonant frequencies are given in the following tables:

Zeroes of $J_m(x)$:

n	j_{0n}	j_{1n}	j_{2n}	j_{3n}	j_{4n}	j_{5n}
	-	0	0	0	0	0
1	2,40482	3,83171	5,13562	6,38016	7,58834	8,77148
2	5,52007	7,01559	8,41724	9,76102	11,06471	12,33860
3	8,65372	10,17347	11,61984	13,01520	14,37254	15,70017
4	11,79153	13,32369	14,79595	16,22347	17,61597	18,98013
5	14,93091	16,47063	17,95982	19,40942	20,82693	22,21780

Zeroes of $J'_m(x)$:

n	j'_{0n}	j'_{1n}	j'_{2n}	j'_{3n}	j'_{4n}	j'_{5n}
	0	-	0	0	0	0
1	3,83170	1,84118	3,05424	4,20119	5,31755	6,41562
2	7,01558	5,33144	6,70613	8,01524	9,28240	10,51986
3	10,17346	8,53632	9,96947	11,34592	12,68191	13,98719
4	13,32369	11,70600	13,17037	14,58585	15,96411	17,31284
5	16,47063	14,86359	16,34752	17,78875	19,19603	20,57551

In both tables the lowest zero points (except for the trivial one) have been highlighted, from them the respective fundamental modes of the two classes of modes are calculated.

The equality of the zero points j'_{0n} and j_{1n} indicates that the derivative of the zeroth order Bessel function coincides with the first order Bessel function:

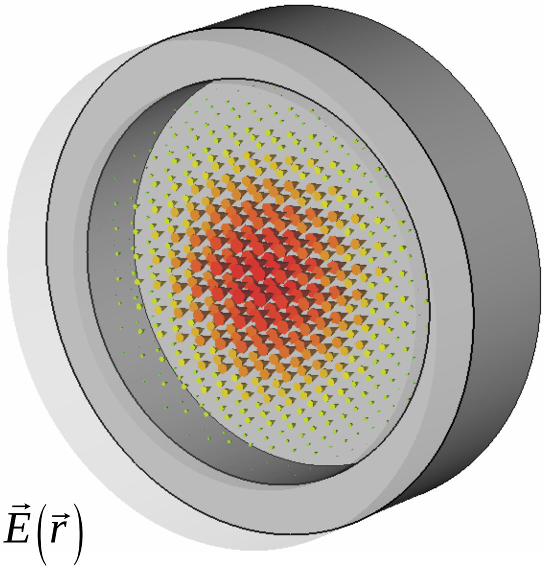
$$\frac{d}{dx} J_0(x) = J_1(x)\,.$$

Therefore the corresponding TE- and TM-Modes have the same resonant frequencies:

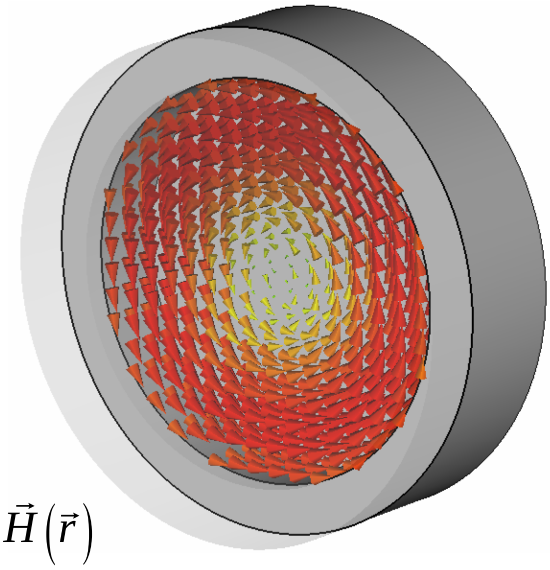
$$\mathrm{TE}_{0np} = \mathrm{TM}_{1np} \quad \text{for arbitrary values of } n \text{ and } p.$$

The following illustrations show a snap-shot of the field distribution of the first two TM-modes of a closed, cylindrical resonator. Further examples can be found in the appendix. Since $p = 0$ there is no dependence on the length of the resonator, the electric field consists of a longitudinal component only. In the figures below, one part of the resonator beyond an arbitrarily chosen section plane is shown with transparency.

TM$_{010}$ mode

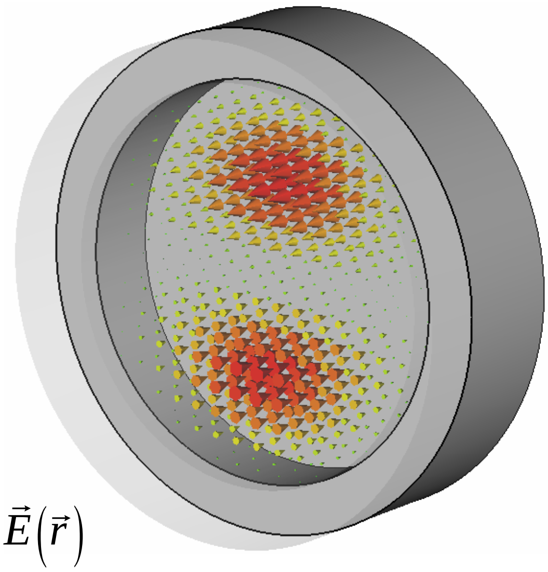


$\vec{E}(\vec{r})$

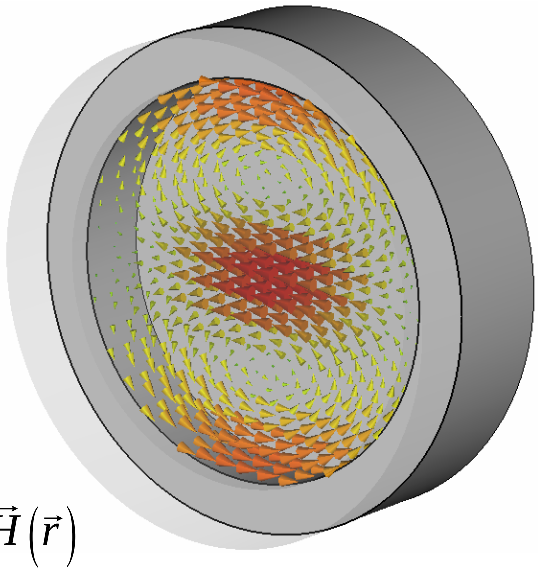


$\vec{H}(\vec{r})$

TM$_{110}$ mode



$\vec{E}(\vec{r})$

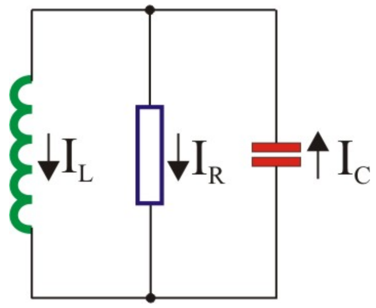


$\vec{H}(\vec{r})$

2. The cavity as an oscillating circuit

2.1. Definition of the characteristic quantities in the unloaded case

We consider the equivalent circuit diagram of a cavity – the LCR-parallel circuit:



We have the usual, well-known relations:

Voltages: $-U_C = U_R = U_L$, $C \cdot \dot{U}_C = I$, $U_L = L \cdot \dot{I}$ (Generator!)

Currents: $I_C = I_R + I_L$

This leads to the following differential equation:

$$\ddot{U} + \frac{1}{RC}\dot{U} + \frac{1}{LC}U = 0$$

We define the following quantities:

- Time constant $\boxed{\tau = R \cdot C}$
- Angular resonant frequency $\boxed{\omega_0 = \frac{1}{\sqrt{L \cdot C}}}$
- Quality factor $\boxed{Q_0 = \frac{2\pi \cdot \text{stored energy}}{\text{losses per period}} = \frac{2\pi \cdot W}{T \cdot P} = \frac{\omega_0 \cdot W}{P}}$

For a weakly damped circuit (oscillating case) one gets:

$$U(t) = U_0 \cdot e^{-\frac{t}{2\tau}} \cdot e^{i(\omega_0 t + \varphi_0)}$$

The stored energy is:

$$W = \frac{1}{2} C \cdot |U|^2 = \frac{1}{2} \cdot e^{-\frac{t}{\tau}} \cdot C \cdot U_0^{\,2},$$

and the loss of energy (dissipated power) is:

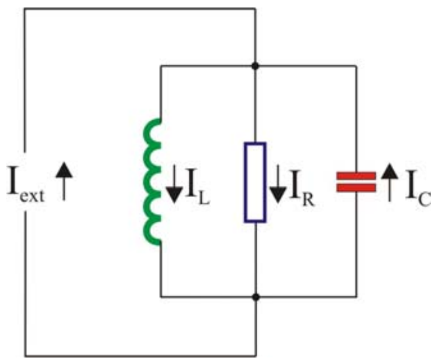
$$P = \dot{W} = -\frac{1}{\tau} \cdot W$$

with this we get the well-known relations for the quality factor:

$$Q = \frac{\omega_0 \cdot W}{P} = \omega_0 \cdot \tau = \omega_0 RC = \frac{R}{\omega_0 L}$$

2.2. Driven oscillations

We use an external current as driving force on the oscillation circuit and get:



Voltages: $-U_C = U_R = U_L$

Currents: $I_C + I_{ext} = I_R + I_L$

$$I_C = \dot{Q}_C = C \cdot \dot{U}_C$$
$$I_R = U_R / R$$
$$\dot{I}_L = U_L / L$$

$$\ddot{U} + \frac{\omega_0}{Q_0} \cdot \dot{U} + \omega_0^{\,2} \cdot U = \frac{1}{C} \dot{I}_{ext}$$

We choose $I_{ext} = \hat{I}_{ext} \cdot e^{i\omega t}$ for the external current. Using the ansatz $U = \hat{U} \cdot e^{i\omega t}$ we get the inhomogeneous complex solution:

$$\hat{U} = \frac{i\omega \hat{I}_{ext} / C}{\omega_0^{\,2} - \omega^2 + \frac{i\omega\omega_0}{Q_0}}$$

After substituting the relation for the quality factor wet gets the following, well-known result:

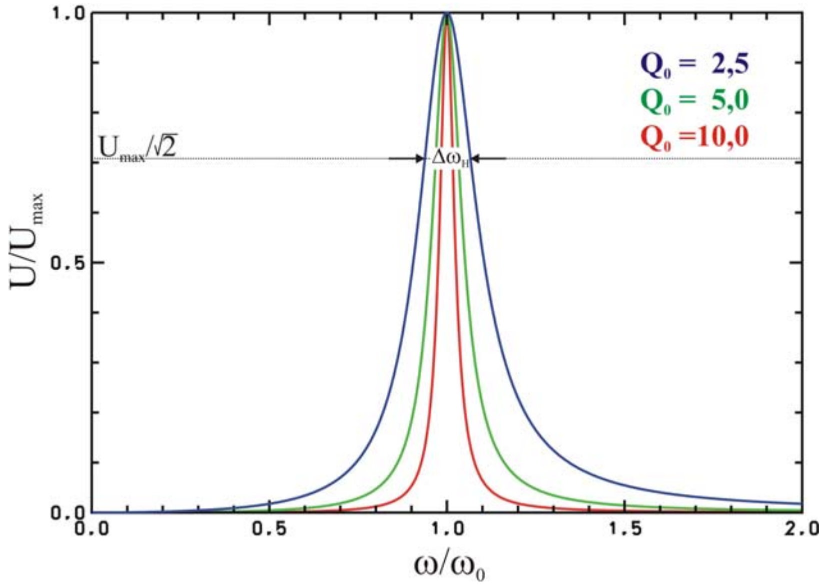
$$\hat{U} = \frac{R \cdot \hat{I}_{ext}}{1 + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \overset{\Delta\omega \ll \omega_0}{\approx} \frac{R \cdot \hat{I}_{ext}}{1 + 2iQ_0 \frac{\Delta\omega}{\omega}}$$

From this, we get the modulus and the phase:

$$\left|\hat{U}\right| = \frac{R \cdot \hat{I}_{ext}}{\sqrt{1 + Q_0^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}} \overset{\Delta\omega \ll \omega_0}{\approx} \frac{R \cdot \hat{I}_{ext}}{\sqrt{1 + 4Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2}}$$

$$\tan\varphi = Q_0\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) \overset{\Delta\omega \ll \omega_0}{\approx} -2Q_0\frac{\Delta\omega}{\omega}$$

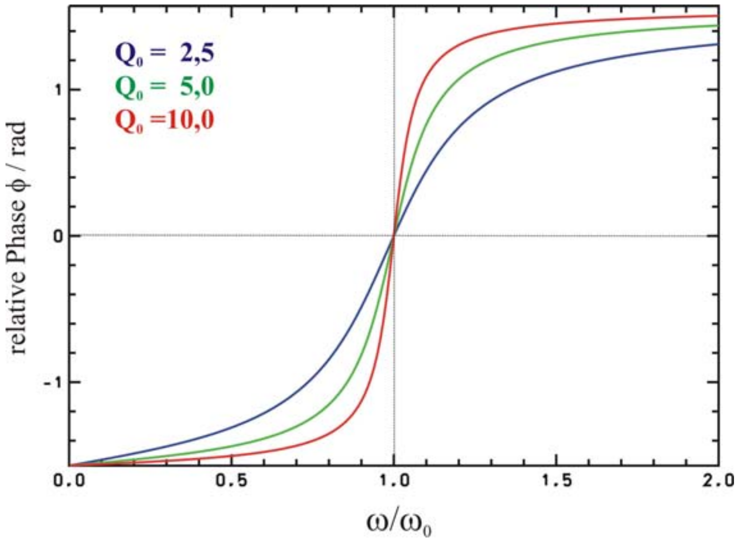
The dependence of the voltage on the frequency is illustrated in the resonance curve. It has the following form (ideal inductor $\Rightarrow$ $\omega_r = \omega_0$!):



The unloaded quality factor can easily be determined by measuring the so-called FWHM (full width at half maximum) $\Delta\omega_H$ from the resonance curve (this can be verified by substitution into the relation for $\left|\hat{U}\right|$):

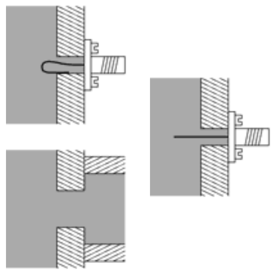
$$Q_0 = \frac{\omega_0}{\Delta\omega_H}, \qquad \Delta\omega_H = \textbf{\underline{full}} \text{ width at half maximum at } \frac{U_{max}}{\sqrt{2}}$$

The phase dependency has a zero at ω_0



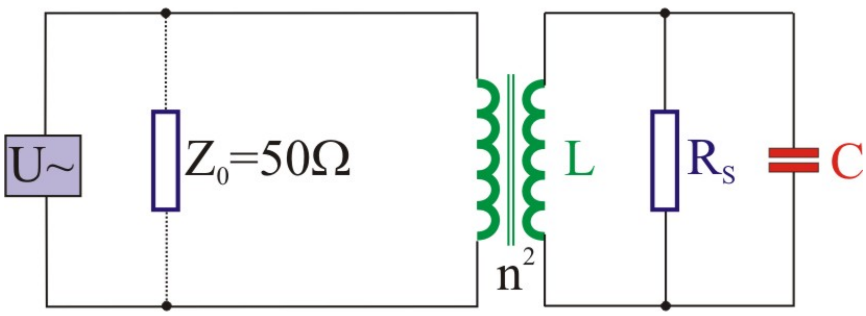
2.3. Loaded case by coupling in of high-frequency

There are basically three different methods how to couple a cavity to high-frequency:



- Coupling to the magnetic field (loop coupling)
- Coupling to the electric field (pin coupling)
- Direct coupling out of a waveguide (hole coupling)

In the following we restrict ourselves to magnetic coupling. The other couplings can be treated in complete analogy. In the case of loop coupling, we have the following scenario in the equivalent circuit;



The purpose of the coupling is to carry the microwaves coming out of the generator to the resonator as complete as possible (without reflections). To achieve this, the transmission line from the generator to the resonator needs to be terminated by its characteristic wave impedance (typically 50Ω). The impedance of the resonator is a complex quantity and only real in the case of resonance. It is then called Shunt-impedance R_S:

$$Z(\omega_0) = R_S = \text{real}$$

The order of magnitude is typically MΩ! Therefore it is transformed down to $Z_a = R_S / n^2$ via loop coupling; in the equivalent circuit this corresponds to a transformer with the turn ratio *n*. The relevant quantity for the reflection is the ration between termination impedance and characteristic impedance. We therefore define the coupling coefficient:

$$\boxed{\kappa = \frac{Z_a}{Z_0} = \frac{R_S}{n^2 \cdot Z_0}}$$

The resonator is additionally loaded by the external transmission line:

$$\frac{1}{R} = \frac{1}{R_S} + \frac{1}{n^2 \cdot Z_0} \quad \Rightarrow \quad \frac{1}{Q} = \frac{1}{Q_0} + \frac{1}{Q_{\text{ext}}}.$$

The unloaded quality factor is reduced to *Q* because of the appearance of an additional external quality factor Q_{ext}. Formally taking the external power dissipation P_{ext} into account, we get the relations:

$$Q_{\text{ext}} = \frac{\omega_0 \cdot W}{P_{\text{ext}}} \quad \Rightarrow \quad Q = \frac{\omega_0 \cdot W}{P + P_{\text{ext}}}$$

$$\kappa = \frac{Q_0}{Q_{ext}} = \frac{P_{ext}}{P} = \frac{R_S}{n^2 \cdot Z_0}$$

We distinguish between three cases:

- $\kappa < 1$: **undercritical coupling**, $Q > Q_0/2$
- $\kappa = 1$: **critical coupling**, $Q = Q_0/2$, **no reflection**!
- $\kappa > 1$: **overcritical coupling**, $Q < Q_0/2$

If the coupling coefficient is known, the unloaded quality factor Q_0 can be calculated from the loaded quality factor Q which was measured:

$$Q_0 = (1+\kappa)\cdot Q$$

3. The complex reflection coefficient

3.1. Dependence on the termination impedance

In the case of reflection we have an incoming ($\hat{U}_+$, $\hat{I}_+$) and a reflected wave ($\hat{U}_-$, $\hat{I}_-$) in the conductor. We define the complex reflection coefficient by the ratio:

$$\boxed{\rho = \frac{\hat{U}_-}{\hat{U}_+}}$$

With termination impedance Z_a and characteristic impedance Z_0 we then have:

$$Z_a = \frac{\hat{U}}{\hat{I}} = \frac{\hat{U}_+ + \hat{U}_-}{\hat{I}_+ + \hat{I}_-}, \qquad Z_0 = \frac{\hat{U}_+}{\hat{I}_+} = \frac{\hat{U}_-}{-\hat{I}_-}$$

By inserting the reflection coefficient ρ_0 at the end of the conductor we get:

$$Z_a = \frac{1+\rho_0}{1-\rho_0}\cdot Z_0 \qquad \Leftrightarrow \qquad \rho_0 = \frac{Z_a - Z_0}{Z_a + Z_0} = \frac{(Z_a/Z_0)-1}{(Z_a/Z_0)+1}$$

3.2. Reflection close to an insulated resonance

In the case of non-overlapping resonances (which we shall consider here exclusively) the impedance of the resonator is known (cp. Chapter 2.2.). Using

$$Z_{\text{Cav}} = \frac{R_S}{1 + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \overset{\Delta\omega \ll \omega_0}{\approx} \frac{R_S}{1 + 2iQ_0\frac{\Delta\omega}{\omega}},$$

as well as the coupling coefficient we get:

$$\frac{Z_a}{Z_0} = \frac{Z_{Cav}}{n^2 \cdot Z_0} = \kappa \frac{Z_{Cav}}{R_S} \overset{\Delta\omega \ll \omega_0}{\approx} \frac{\kappa}{1 + 2iQ_0 \frac{\Delta\omega}{\omega}}.$$

In dependence on the frequency shift $\Delta\omega$ from the resonant frequency ω_0, this yields for the complex reflection coefficient:

$$\rho_0(\Delta\omega) = \frac{\kappa - \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/_{\omega}\right)}{\kappa + \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/_{\omega}\right)}.$$

Of course this is only true directly at the location of the coupling into the resonator. If the reflection coefficient is measured at a distance *l* to the coupling (this is due to the presence of a transmission line between the position where the measurement takes place and the coupling), twice the delay factor of the wave in the line is added. With the wave number

$$k = \frac{\omega}{\mathrm{v}_{ph}} = \omega \cdot \sqrt{L' \cdot C'},$$

which depends on the properties of the conductor (inductivity *L*' and. Capacity *C*' per length) **and on the frequency (!)**,neglecting losses on the conductor we get:

$$\rho(\Delta\omega) = \rho_0(\Delta\omega) \cdot e^{-2ikl} = \frac{\kappa - \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/_{\omega}\right)}{\kappa + \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/_{\omega}\right)} \cdot e^{-2ikl}.$$

(Note that we have to be careful with waveguides, since everything depends on their excited mode)

In consideration of the large quality factors ($Q_0 = 1000 - 10000$) we will neglect this effect for the time being – for larger frequency ranges this can be observed quite nicely on the oscilloscope during the vectorial measurement of the reflection coefficient and leads to circles with almost constant radius, which can be used for normalization in the analysis.

4. Measurement of $|\rho|$

4.1. The „resonance curve“

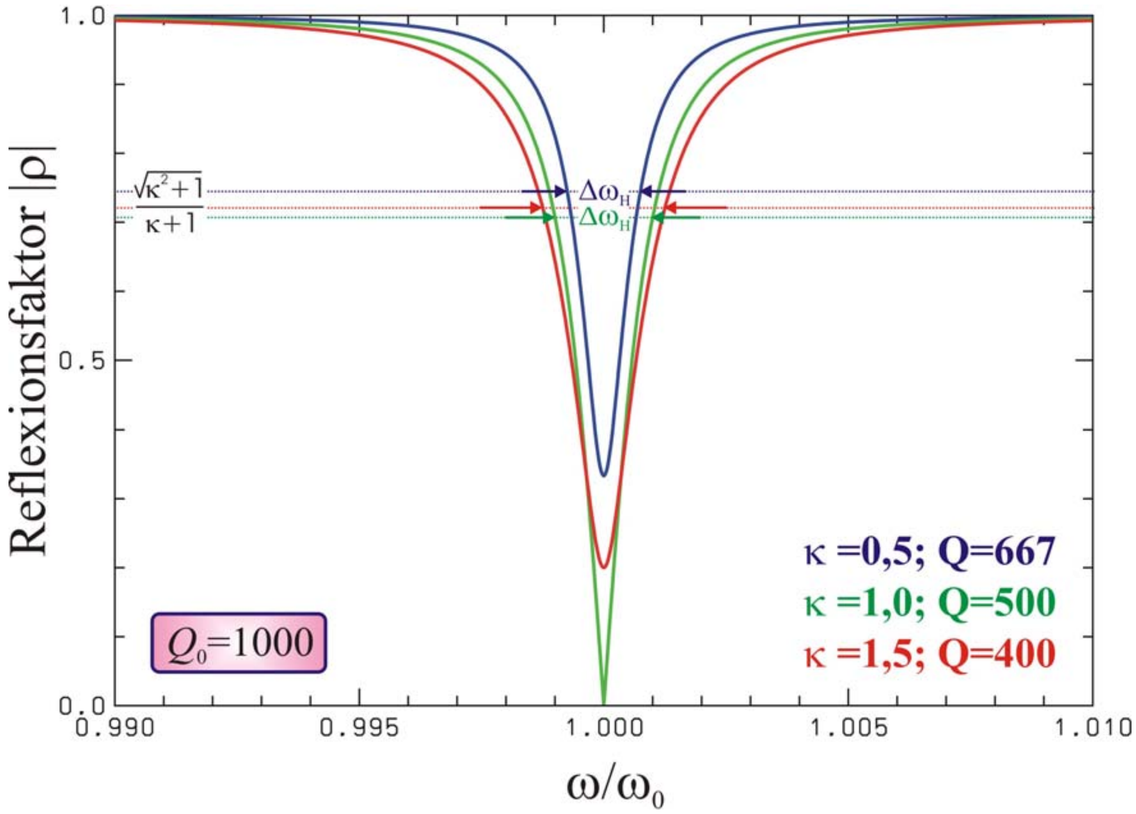
By separating real and imaginary part we get for the complex reflection coefficient directly at the coupling:

$$\rho_0(\Delta\omega) = \frac{\left(\kappa^2 - 1\right) - 4Q_0^2\left(\Delta\omega/\omega\right)^2 - 4i\kappa Q_0 \,\Delta\omega/\omega}{\left(\kappa + 1\right)^2 + 4Q_0^2\left(\Delta\omega/\omega\right)^2}.$$

And its modulus:

$$|\rho(\Delta\omega)| = |\rho_0(\Delta\omega)| = \sqrt{\frac{\left(\kappa - 1\right)^2 + 4Q_0^2\left(\Delta\omega/\omega\right)^2}{\left(\kappa + 1\right)^2 + 4Q_0^2\left(\Delta\omega/\omega\right)^2}}.$$

On a scalar network analyser on then sees the following picture of the reflection coefficient (and this is exact since the length of a conductor does not contribute if it operates without losses):



4.2. Determination of the resonant frequency and the coupling

At resonance ($\Delta\omega = 0$), the reflection is minimal. The resonant frequency ω_0 is read off by finding the minimum of the reflection coefficient with the cursor functions.

$$\left|\rho(\Delta\omega = 0)\right| = \left|\frac{\kappa - 1}{\kappa + 1}\right| \quad \Rightarrow \quad \kappa = \begin{cases} (1+|\rho|)/(1-|\rho|), & \rho > 0 \\ (1-|\rho|)/(1+|\rho|), & \rho < 0 \end{cases}$$

Unfortunately it is not possible to distinguish between $\rho > 0$ and $\rho < 0$.

Note: A precise measurement of $|\rho|$ requires a calibrated analyzer!!

4.3. Determination of the quality factor

The loaded quality factor is

$$Q = \frac{Q_0}{1+\kappa} = \frac{\omega_0}{\Delta\omega_H} \quad \overset{\Delta\omega_H = 2\Delta\omega}{\Rightarrow} \quad \frac{\Delta\omega}{\omega} \approx \frac{1+\kappa}{2Q_0}$$

because the corresponding frequency shift is only half the FWHM! With this we get for the reflection coefficient:

$$\left|\rho\left(\Delta\omega_H/2\right)\right| = \sqrt{\frac{(\kappa-1)^2+(\kappa+1)^2}{(\kappa+1)^2+(\kappa+1)^2}} = \frac{\sqrt{\kappa^2+1}}{\kappa+1}$$

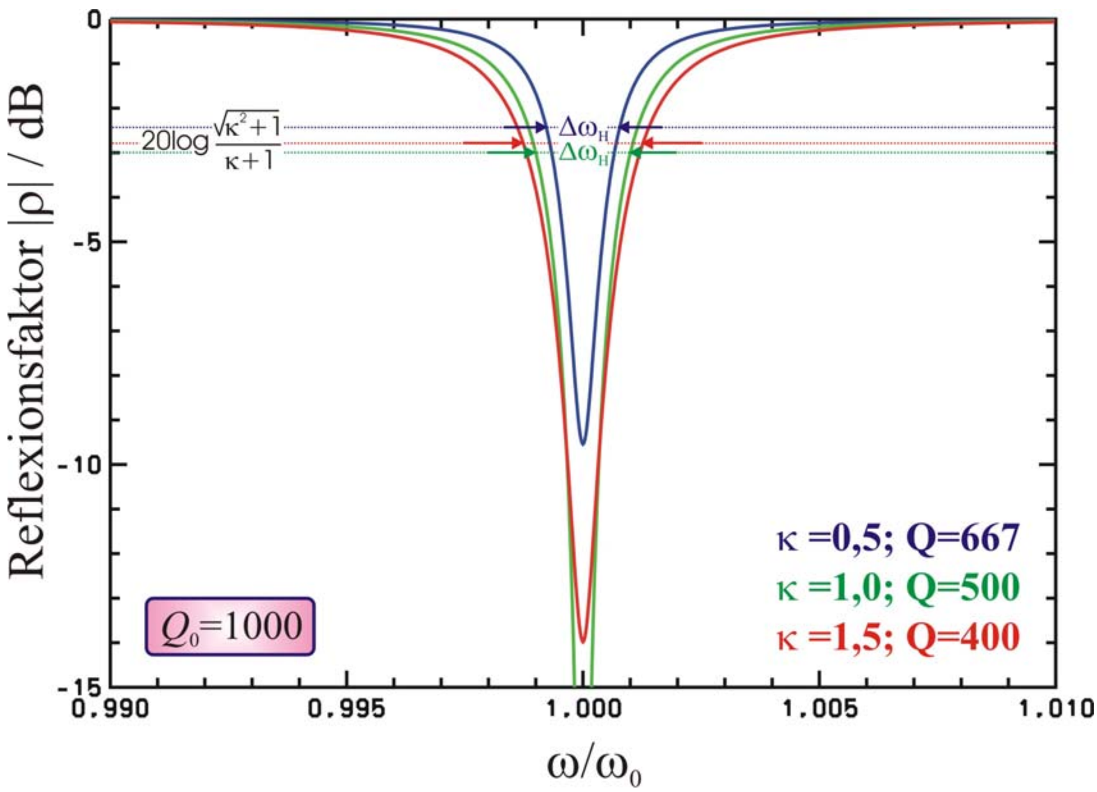
It is therefore important to keep in mind that

only in the case κ=1 the full 3dB-FWHM –corresponding to $\rho = 1/\sqrt{2}$ because of the definition of the dB-values (dB = 20·log(U/U$_0$))– is taken to determine the quality factor, in all other cases it has to be taken at

$$\left|\rho\left(\Delta\omega_H/2\right)\right| = \frac{\sqrt{\kappa^2+1}}{\kappa+1} \neq \frac{1}{\sqrt{2}}$$

(compare the lin/log-diagrams)!

A resonance with the quality factor Q_0=1000 and different coupling coefficients (κ = 0,5; 1; 1,5) plotted logarithmically looks as follows:



5. Vectorial measurement of the reflection coefficient

5.1. The „resonance curve“ in the complex plane

For the complex reflection coefficient we had the following expression:

$$\boxed{\rho(\Delta\omega) = \frac{\left(\kappa^2-1\right)-4Q_0^2\left(\Delta\omega/\omega\right)^2 - 4i\kappa Q_0 \,\Delta\omega/\omega}{\left(\kappa+1\right)^2+4Q_0^2\left(\Delta\omega/\omega\right)^2}\cdot e^{-2ikl}}$$

If we neglect the delay coefficient e^{-2ikl} for now and plot the reflection coefficient in the complex plane, then (close to the resonance) ρ_0 describes a circle of radius r around (x_0, y_0)

$$\boxed{x_0 + i\,y_0 = -\frac{1}{1+\kappa}, \qquad r = \frac{\kappa}{1+\kappa}}$$

This can be verified in a rather lengthy calculation by plugging in r and x_0. Thus we will only show some intermediate steps:

$$\underbrace{\left[\frac{\left(\kappa^2-1\right)-4Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2}{\left(\kappa+1\right)^2+4Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2}+\frac{1}{1+\kappa}\right]^2}_{(x-x_0)^2} + \underbrace{\left[\frac{4\kappa Q_0\frac{\Delta\omega}{\omega}}{\left(\kappa+1\right)^2+4Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2}\right]^2}_{y^2} = \underbrace{\left[\frac{\kappa}{1+\kappa}\right]^2}_{r^2}$$

$$\cdots$$

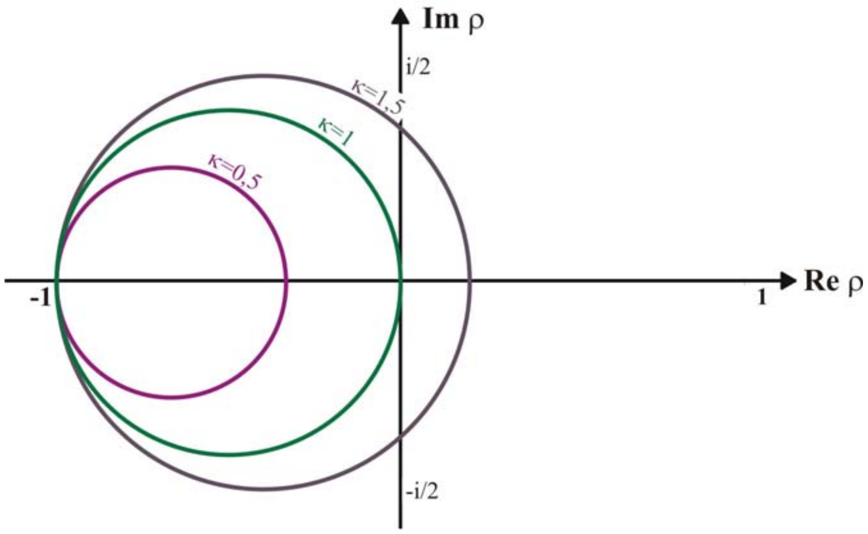
$$\Leftrightarrow \left[\frac{\kappa\left(1+\kappa\right)-\frac{4\kappa Q_0^2\left(\Delta\omega/\omega\right)^2}{1+\kappa}}{\left(1+\kappa\right)^2+4Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2}\right]^2 + \frac{16\kappa^2Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2}{\left[\left(1+\kappa\right)^2+4Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2\right]^2} = \frac{\kappa^2}{\left(1+\kappa\right)^2}$$

$$\cdots$$

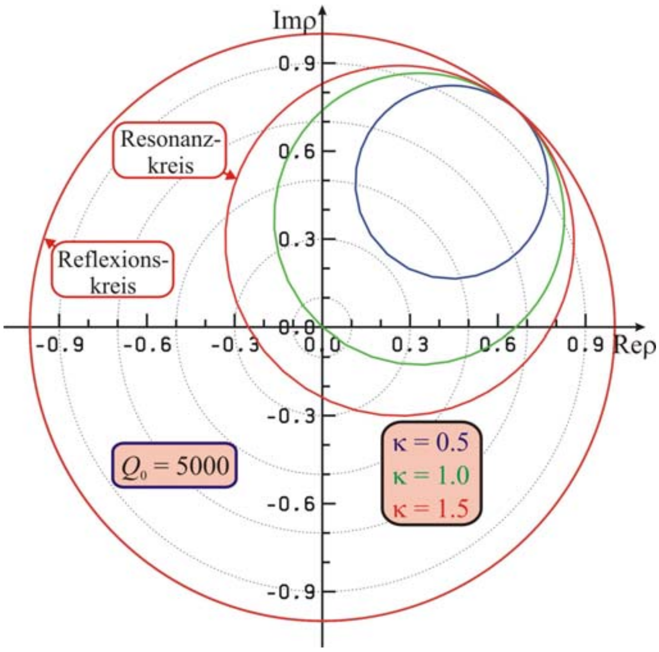
$$\Leftrightarrow \left[\kappa\cdot\frac{\left(1+\kappa\right)+\frac{4Q_0^2\left(\Delta\omega/\omega\right)^2}{1+\kappa}}{\left(1+\kappa\right)^2+4Q_0^2\left(\frac{\Delta\omega}{\omega}\right)^2}\right]^2 = \frac{\kappa^2}{\left(1+\kappa\right)^2}$$

q.e.d.

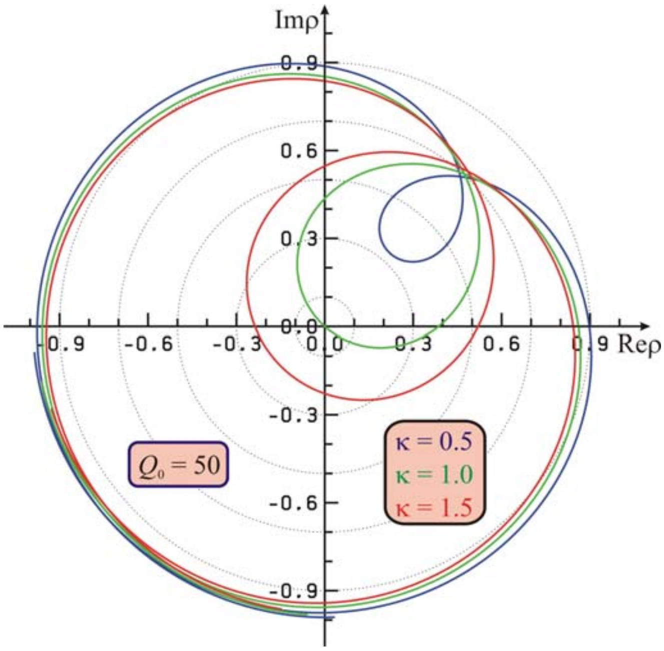
Thus radii und positions of the circles depend on the coupling coefficient but not at all on the quality factor! If the delay coefficient is neglected, all these circle go through (-1; 0), and we get:



The delay coefficient rotates the circles around the origin. In the case of large quality factors the change in the shape of the circles due to the delay coefficient is negligible. We get, for instance, the following picture where we can see the actual "resonance circle" and the "reflection circle" with radius one which is generated by the delay coefficient:



In the case of lower quality factors the circles are deformed by the delay coefficient. Since the loaded quality factor depends on the coupling coefficient the deformation also depends on the coupling, e.g. for $\omega_0 = 3\text{GHz}$, $l = 2\text{m}$:



No being able to calibrate the vectorial measurement with the available set-up (phase discriminator, calibration is only possible with rather expensive vectorial analyzers), we have to neglect the effects of the delay coefficient. For the determination of the characteristic quantities these effects are small due to the large quality factors and can be further reduced by shortening the lines if necessary.

5.2. Determination of the resonant frequency and the coupling

Before the reflection coefficient can be read off from the oscilloscope, the origin of the Smith-Diagram which is displayed has to be matched with the one on the scale of the oscilloscope (calibration)

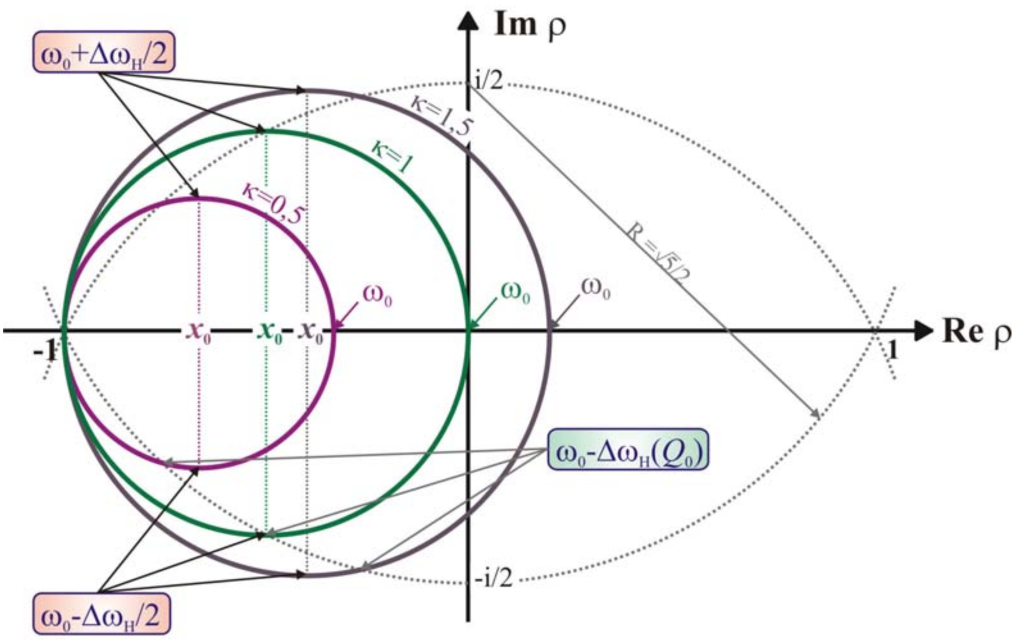
There are basically two ways how to achieve this:

- We create a state without reflection with the 50 Ω-terminator and thus get the point where $\rho = 0$ on the oscilloscope. In practice the termination is never entirely without reflection, it is therefore recommended to do the calibration at the resonant frequency and to reduce the frequency range.

- We look at the state of complete reflection i.e. $\rho = 1$. It can then be centred on the oscilloscope using the circular marks.

To determine the reflection coefficient in resonance we proceed as follows: The rotation of the circles around the origin due to the delay coefficient has to be compensated be rotating the coordinate system. The scale of the oscilloscope can be rotated around the origin. It must be set in such a way that the point of intersection between resonance circle and reflection circle is on the real axis.

In the case of resonance the impedance of the resonator is real; the curve of the complex reflection coefficient thus has to cross the real axis. The corresponding frequency is the resonant frequency ω_0.



The reflection coefficient in resonance ρ_0 can be determined as follows:

First we read off the distance *d* (depending on the coupling it carries a sign) between the resonance point and the origin (which corresponds to ρ_0). Then we have to determine the radius *R* of the reflection circle (corresponds to $\rho = 1$) which is needed for normalization It is sufficient to do all this in units of the oscilloscope scale. The reflection coefficient is simply the ratio between the two:

$$\rho_0 = d/R\,.$$

The coupling coefficient is then calculated as

$$\kappa \;=\; (1+\rho_0)/(1-\rho_0)$$

5.3. Determination of the quality factor

For the loaded quality factor we have – in analogy to the scalar measurement –

$$Q \;=\; \frac{Q_0}{1+\kappa} \;=\; \frac{\omega_0}{\Delta\omega_H} \quad \overset{\Delta\omega_H = 2\Delta\omega}{\Rightarrow} \quad \frac{\Delta\omega}{\omega} \approx \frac{1+\kappa}{2Q_0}$$

By plugging this in we get for the reflection coefficient

$$\rho_0(\pm\Delta\omega_H/2) \;=\; \frac{(\kappa^2-1)-(\kappa+1)^2 \mp 2i\kappa(\kappa+1)}{(\kappa+1)^2+(\kappa+1)^2}$$

From this we get by comparison with the centre of the circle (x_0;0) and radius *r*:

$$\boxed{\rho_0(\pm\Delta\omega_H/2) \;=\; -\frac{1}{\kappa+1} \mp i\frac{\kappa}{\kappa+1} \;=\; x_0 \mp i\cdot r}$$

To determine the FWHM the resonance circle is centred on the origin of the coordinate system and the frequency range between the upper and lower intersection with the imaginary axis is measured.

From the vectorial diagram the unloaded quality factor Q_0 is readily read of. By plugging the relation $Q_0 = \omega_0 / \Delta\omega$ (this time with the full frequency shift!) into the formula for ρ, we get:

$$\rho_0(\pm\Delta\omega) = \frac{\kappa^2 - 5}{(\kappa+1)^2 + 4} \mp i \frac{4\kappa}{(\kappa+1)^2 + 4}$$

The κ-dependence of these values describes a circle with radius $r = \sqrt{5}/2$ around $(0; i/2)$ and $(0; -i/2)$ respectively, which can again by verified simply by plugging in, e.g. for $y_0 = -1/2$:

$$\left[\kappa^2 - 5\right]^2 + \left[-4\kappa + \frac{1}{2}\left((\kappa+1)^2 + 4\right)\right]^2 = \frac{5}{4} \cdot \left[(\kappa+1)^2 + 4\right]^2$$

$$\Leftrightarrow \qquad \kappa^4 - 10\kappa^2 + 25 + \frac{\kappa^4 + 20\kappa^3 + 110\kappa^2 + 100\kappa + 25}{4} = \frac{5}{4} \cdot \left[\kappa^4 + 4\kappa^3 + 14\kappa^2 + 20\kappa + 25\right]$$

$$\Leftrightarrow \qquad \frac{5\kappa^4 + 20\kappa^3 + 70\kappa^2 + 100\kappa + 125}{4} = \frac{5}{4} \cdot \left[\kappa^4 + 4\kappa^3 + 14\kappa^2 + 20\kappa + 25\right] \qquad \text{q.e.d.}$$

We therefore get the frequency shift $\Delta\omega$ from the resonant frequency ω_0 and thus the FWHM (needed for the determination of the unloaded quality factor) from the intersection of the resonance circle with one of the mentioned circles around $(0; \pm i/2)$. The unloaded quality factor is then calculated according to $Q_0 = \omega_0 / \Delta\omega$.

5.4. Other important quantities and the Smith-Diagram

Any microwave conductor has a characteristic impedance Z_0 and is terminated by some impedance Z_a at its end.When the termination impedance is transformed along the conductor (which is formally equivalent to a multiplication with the corresponding delay coefficient) it turns out that it becomes real at certain places. One therefore likes to talk of a real termination impedance which can be transformed along the conductor into the desired complex impedance if necessary. In this context we define the

Impedance matching $$m = \frac{1-|\rho|}{1+|\rho|} = \begin{cases} \left|\frac{Z_a}{Z_0}\right| = \kappa, & \text{für } Z_a < Z_0 \\ \left|\frac{Z_0}{Z_a}\right| = \frac{1}{\kappa}, & \text{für } Z_a > Z_0 \end{cases}$$

The propagation of microwaves causes reflections for $Z_a \neq Z_0$ and the following three quantities are of special interest:

1. **Relative norm. resistance** $z_s = \left|\frac{Z(s)}{Z_0}\right|$ **at the point s on the conductor**
2. **Reflection coefficient** $$\rho = \frac{Z_s - Z_0}{Z_s + Z_0} = \rho_0 \cdot e^{-i2ks} = \frac{\kappa - 1}{\kappa + 1} \cdot e^{-i2ks}$$
3. **Standing wave ratio** $$S = \frac{|\hat{U}_{max}|}{|\hat{U}_{min}|} = \frac{1+|\rho|}{1-|\rho|} = \frac{1}{m}$$

Other quantities can easily be calculated from these three. For instance the effective power converted at the termination impedance, P_A (i.e. in our case the power dissipated in the resonator) depends on the supplied power P_0 as:

$$\boxed{P_a = \frac{4S}{(1+S)^2} \cdot P_0}$$

This can be calculated by inserting $P_0 = P_+ = \frac{1}{2}|\hat{U}_+|^2 \big/ |Z_0|$ and $P_- = \frac{1}{2}|\hat{U}_-|^2 \big/ |Z_0|$

into $P_a = P_+ - P_-$.

For the transformation of the (relative) impedances along the conductor the following additional lines are included in the diagram of the reflection coefficient in the complex plane:

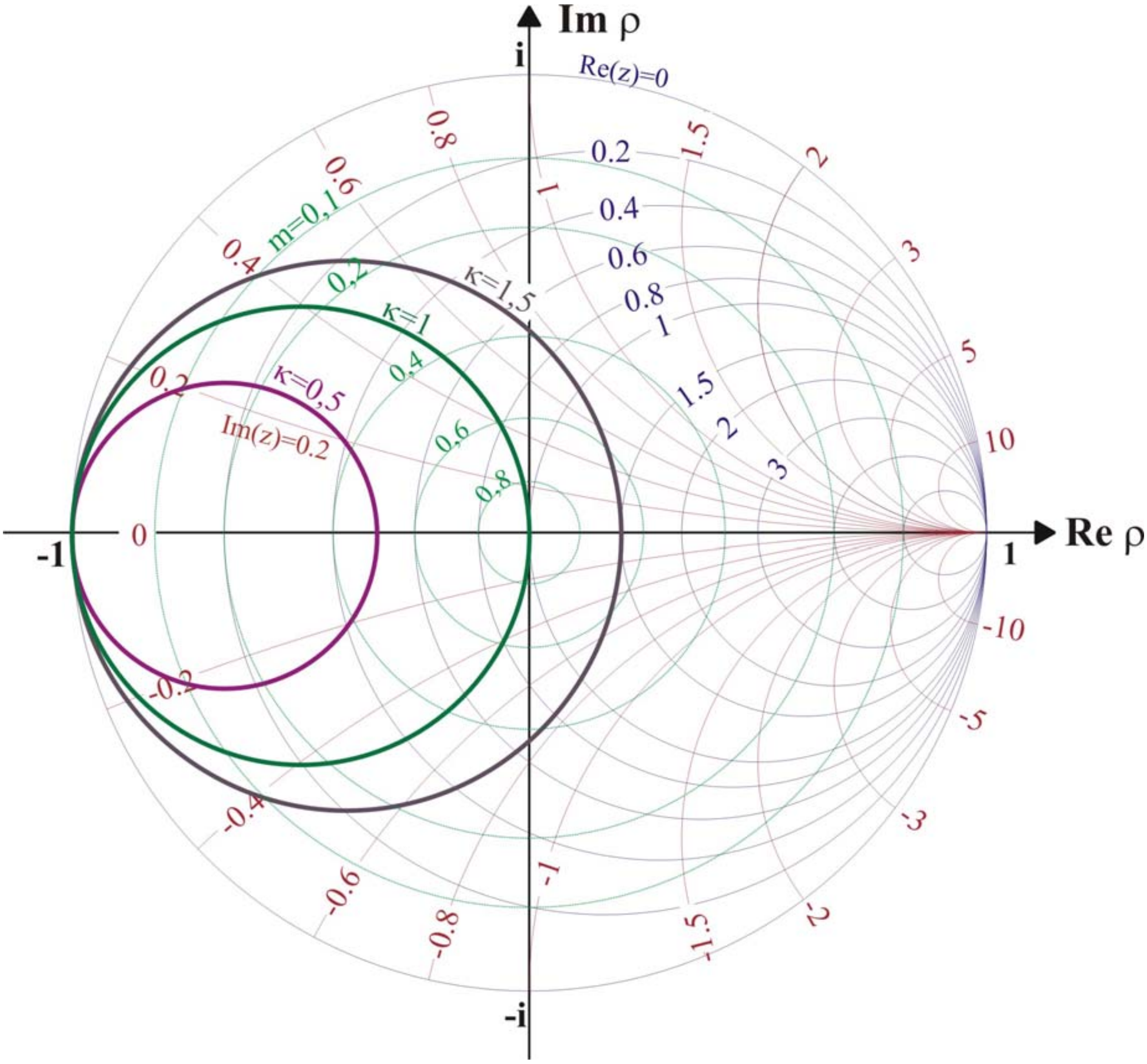
- **m –circles**: $m = \text{konst.} \leftrightarrow |\kappa| = \text{konst.}$
- **“real part circles”**: $\mathrm{Re}(z_s) = \mathrm{Re}(Z_s/Z_0) = \text{konst.}$

- **“imaginary part circles”**: $\mathrm{Im}(z_s) = \mathrm{Im}(Z_s/Z_0) = \text{konst.}$
- ***l* – „circles“**: $e^{-i2ks} = \text{konst.}$

the following radii and centre points can easily be verified by plugging the relation for the complex reflection coefficient into the corresponding equations above:

- **m – circles**: $M = x_0 + i\,y_0 = 0$, $r = \dfrac{1-m}{1+m}$
- **“real part circles”**: $M = x_0 + i\,y_0 = \dfrac{\mathrm{Re}(\kappa)}{\mathrm{Re}(\kappa)+1}$, $r = \dfrac{1}{\mathrm{Re}(\kappa)+1}$
- **“imaginary part circles”**: $M = x_0 + i\,y_0 = 1 + \dfrac{i}{\mathrm{Im}(\kappa)}$, $r = \dfrac{1}{\mathrm{Im}(\kappa)}$
- ***l* – „circles“**: lines through $x_0 + i\,y_0 = 0$ and $x_1 + i\,y_1 = e^{-i2ks}$

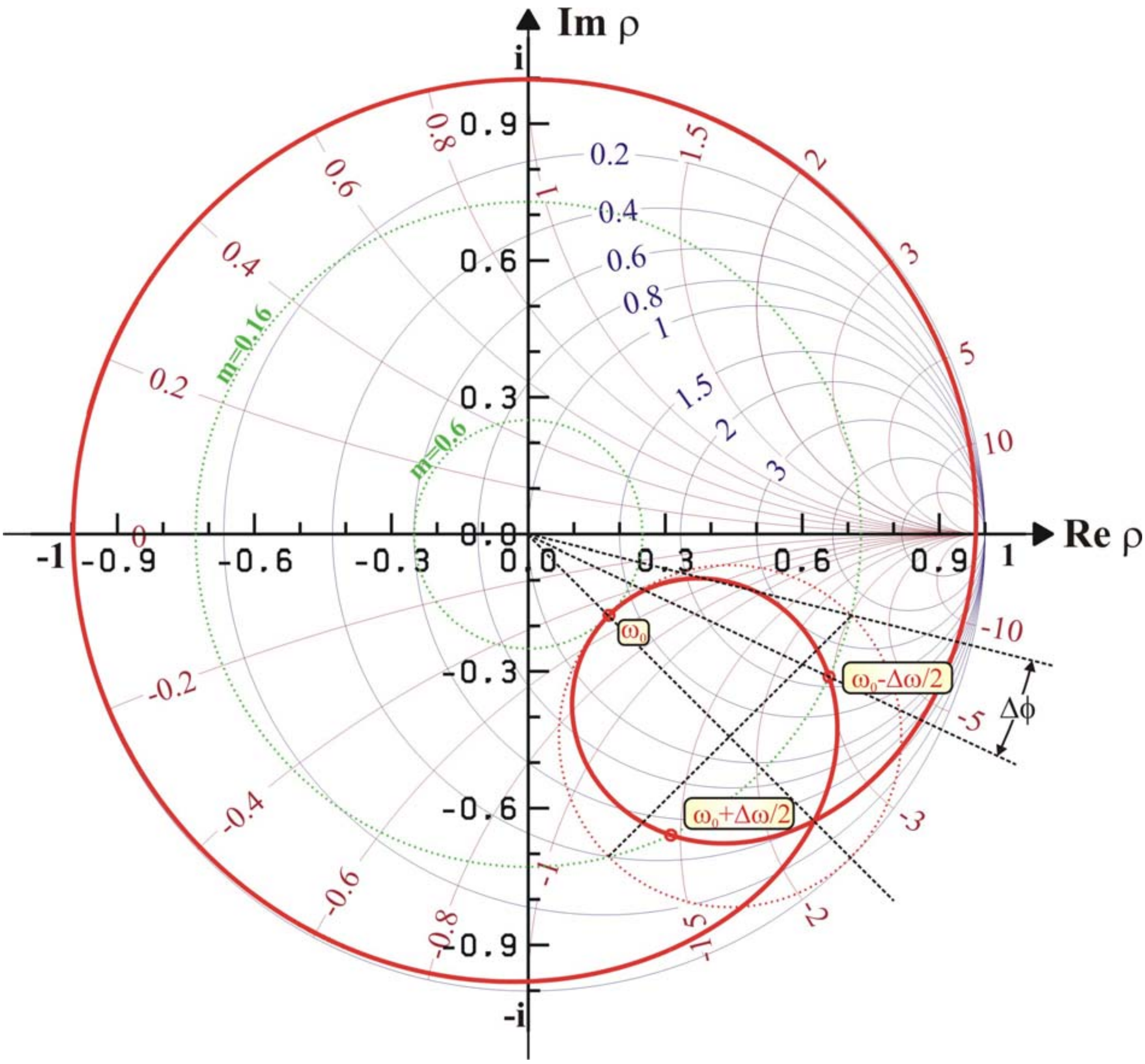
This enhanced diagram is called „**Smith-Diagram**“and in our case it allows the coupling coefficient to be read off very easily (as the intersection point of the resonance circle with the real part circle):



Using this diagram, resonant frequency, coupling coefficient, quality factor and impedance of the resonator can be determined, also taking the delay coefficient into account. As this allows for a deeper insight into possible sources of error, we shall consider the following, showcase

- Resonant frequency: $\omega_0 = 2\pi \cdot 3\,\mathrm{GHz}$,
- Unloaded quality factor: $Q_0 = 500$,
- Coupling coefficient: $\kappa = 0{,}6$,
- Length of the conductor: $l = 981{,}25\,\mathrm{mm} \mathrel{\hat{=}} \left(9 + 13/16\right) \cdot \lambda_0$

and make the simplifying assumption that the wavelength in the conductor equals the vacuum wavelength. On the oscilloscope we get the following picture:



From the intersection with the m – circle we read of the coupling coefficient $\kappa = 0{,}6$ (for overcritical coupling the value $1/m$ has to be used) and determine the corresponding resonant frequency $\nu_0 = 3\,\mathrm{GHz}$ using the frequency generator. The angle by which the centre of the resonance circle is rotated around the origin gives us the length of the conductor expect for possible multiples of $\lambda/2$ ($\lambda/2$ corresponds to one full rotation); with $\Delta\varphi = 225^\circ$ we therefore have $l = 5/8 \cdot \lambda/2 \mod(\lambda/2)$. For determining the quality factor the deformation of the circle by the delay coefficient has to be taken into account. First we draw the ideal resonance circle around the "centre" of the real resonance circle. We afterwards determine the intersection points with the diameter (of the ideal circle) perpendicular to the line between centre and origin. Then we transform these points along the intersecting m – circle to the real resonance circle. At this point we read off the frequency shift and from the FWHM get the loaded quality factor $Q = 312{,}5$.

> Here we can see quite nicely that in the case of moderate quality factors and long conductors the determination of Q from the intersection of the diameter with the real resonance circle leads to values for $\Delta\omega$ which are too small!

We also learn that the m–value ($m \approx 0{,}16$) at $\omega_0 \pm \Delta\omega/2$ results in a standing wave ratio of $S = 1/m \approx 6{,}3$ and we still feed about half of the power into the resonator.

The unloaded quality factor arises as $Q_0 = (1+\kappa)\cdot Q = 1{,}6 \cdot Q = 500$.

Furthermore, the shift of the points at $\omega_0 \pm \Delta\omega/2$ due to the delay allows the complete determination of the length of the conductor. We draw the corresponding connection lines between the origin and the points $\omega_0 \pm \Delta\omega/2$ on the ideal as well as on the real resonance circle and get an angle of $\Delta\phi = 11{,}3^\circ$. From this arises with

$$\Delta\phi = -2\Delta kl = -\frac{4\pi l}{\Delta\lambda} = -\frac{4\pi l}{\lambda_0}\cdot\frac{\lambda_0}{\Delta\lambda} = \frac{4\pi l}{\lambda_0}\cdot\frac{\Delta\omega}{\omega_0} = \frac{2\pi l}{\lambda_0}\cdot\frac{1+\kappa}{Q_0}$$

$$\Leftrightarrow \qquad \boxed{l = \frac{\Delta\phi}{2\pi}\cdot\frac{Q_0}{1+\kappa}\cdot\lambda_0 \approx 9{,}81\cdot\lambda_0}$$

which fits nicely with the actual situation!

6. Bead pull measurements

For measuring the electric and magnetic fields within the resonator (preferably on the axis) antennas are unsuitable because the necessary cables would alter the field distribution in the resonator. Instead a small impurity (a dielectric or conducting object) which distorts the field slightly is inserted into the resonator.
This perturbation causes a shift in the resonant frequency ($\omega_0 \to \omega$) and at constant excitation with ω_0 a change in the reflection coefficient. Both can be measured and used for calculating the fields. The case where the shift of the resonant frequency is measured is called the resonant method since the resonator continues to be driven in resonance. In the non-resonant method on measures the change in the reflection coefficient without any change in the excitation
For the determination the shunt impedance it is sufficient to measure the electric field. This is done by means of a small dielectric impurity which will now be treated in some more detail.

6.1. Slater formula

In the following we distinguish between the unperturbed fields (i.e. without the impurity in the cavity)

$$\vec{E}_0 \cdot e^{i\omega_0 t} \quad \text{und} \quad \vec{H}_0 \cdot e^{i\omega_0 t},$$

which appear upon excitation with the original resonant frequency ω_0, and the perturbed fields (i.e. the impurity is within the cavity)

$$\vec{D} \cdot e^{i\omega t} = \left(\varepsilon_0 \vec{E}_0 + \vec{P}\right) \cdot e^{i\omega t} \quad \text{und} \quad \vec{B} \cdot e^{i\omega t} = \left(\mu_0 \vec{H}_0 + \vec{M}\right) \cdot e^{i\omega t},$$

which are excited with the modified resonant frequency ω. The additional polarisation $\vec{P}$ and magnetisation $\vec{M}$ are due to the impurity.

In the following we calculate the change in the stored energy. To do this, we start with the Maxwell-Equation in the interior of the resonator where, because of $\vec{j} = 0,\ \rho = 0$, we get:

$$\vec{\nabla} \times \vec{H}_0 = \varepsilon_0 \frac{\partial \vec{E}_0}{\partial t} = i\omega_0 \varepsilon_0 \vec{E}_0 \qquad \rightarrow \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = i\omega \vec{D}$$

$$\vec{\nabla} \times \vec{E}_0 = -\mu_0 \frac{\partial \vec{H}_0}{\partial t} = i\omega_0 \mu_0 \vec{H}_0 \qquad \rightarrow \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

Multiplying the first equation with $\vec{E}_0^*$ and the second with $\vec{H}_0^*$, applying the operator identity $\vec{a}\bullet\left(\vec{\nabla} \times \vec{b}\right) = \vec{\nabla}\bullet\left(\vec{a} \times \vec{b}\right) + \vec{b}\bullet\left(\vec{\nabla} \times \vec{a}\right)$ and substituting the curl of the unperturbed fields by their time derivative according to the equations on the left yields:

$$\vec{\nabla}\bullet\left(\vec{H} \times \vec{E}_0^*\right) - i\omega_0\mu_0 \vec{H}_0^*\bullet\vec{H} = i\omega \vec{E}_0^*\bullet\vec{D}$$

$$\vec{\nabla}\bullet\left(\vec{E} \times \vec{H}_0^*\right) + i\omega_0\varepsilon_0 \vec{E}_0^*\bullet\vec{E} = i\omega \vec{H}_0^*\bullet\vec{B}$$

We integrate over the interior of the resonator and apply gauss' theorem to get:

$$\underbrace{\oint\!\!\!\oint_{\partial V} \left(\vec{H} \times \vec{E}_0^*\right)\bullet d\vec{A}}_{=0} - i\omega_0\mu_0 \iiint_V \left(\vec{H}_0^*\bullet\vec{H}\right) dV = i\omega \iiint_V \left(\vec{E}_0^*\bullet\vec{D}\right) dV$$

$$\overbrace{\oint\!\!\!\oint_{\partial V} \left(\vec{E} \times \vec{H}_0^*\right)\bullet d\vec{A}} + i\omega_0\varepsilon_0 \iiint_V \left(\vec{E}_0^*\bullet\vec{E}\right) dV = i\omega \iiint_V \left(\vec{H}_0^*\bullet\vec{B}\right) dV$$

The surface integral vanishes because of the boundary condition on the conducting walls of the cavity. Plugging in the perturbed fields yields:

$$-\omega_0\mu_0 \iiint_V \left(\vec{H}_0^*\bullet\vec{H}\right) dV = \omega\varepsilon_0 \iiint_V \left(\vec{E}_0^*\bullet\vec{E}\right) dV + \omega \iiint_V \left(\vec{E}_0^*\bullet\vec{P}\right) dV$$

$$\omega_0\varepsilon_0 \iiint_V \left(\vec{E}_0^*\bullet\vec{E}\right) dV = -\omega\mu_0 \iiint_V \left(\vec{H}_0^*\bullet\vec{H}\right) dV - \omega \iiint_V \left(\vec{H}_0^*\bullet\vec{M}\right) dV$$

We multiply the first of the equations with ω, the second one with ω_0, subtract them from each other and in the approximation of a large quality factor ($\omega\omega_0 \approx \omega^2$) and a small volume of the impurity ($\vec{E}_0^* \bullet \vec{E} \approx |E|^2$)

$$\frac{\omega_0^2 - \omega^2}{\omega^2} = \frac{\iiint\limits_{V_S} \left(\vec{E}_0^* \bullet \vec{P} - \vec{H}_0^* \bullet \vec{M} \right) dV}{\varepsilon_0 \iiint\limits_{V} |E_0|^2 \, dV} \approx 2 \frac{\Delta\omega}{\omega_0}$$

Now we only have to integrate over the volume V_S of the impurity, because only within it the magnetisation and polarisation are different from zero. In the denominator of the so-called Slater formula there is twice the energy stored in the resonator, which can be calculated by using its relation to the quality factor and the resonance frequency, cf. chapter 2.1 and chapter 2.3 respectively.

6.2. Resonant bead pull measurement

In the case of a spherical dielectric impurity the polarisation is parallel and proportional to the electric field. If it has a small dielectric permittivity we have in a good approximation:

$$\vec{P} = (\varepsilon - \varepsilon_0) \cdot \vec{E}_0, \qquad \vec{M} = 0$$

We define the perturbing constant as

$$\alpha_S = \tfrac{1}{2} \cdot (\varepsilon - \varepsilon_0) \cdot V_S$$

and get the electric field strength $E_0(z)$ at the point z on the axis in dependence on the shift in the resonant frequency $\Delta\omega(z)$ measured there.

$$E_0(z) = \sqrt{2 \cdot \frac{W}{\alpha_S} \cdot \frac{\Delta\omega(z)}{\omega_0}}$$

6.3. Non-resonant bead pull measurement

If we continue to excite the resonator (including the impurity) with the frequency ω_0 (which has a difference of $\Delta\omega$ from its resonant frequency), the reflection coefficient changes. Using the knowledge obtained in chapter four, we get:

$$\rho_0(\omega_0) = \frac{\kappa - 1}{\kappa + 1}, \qquad \rho(\omega) = \frac{\kappa - \left(1 + 2iQ_0 \Delta\omega / \omega_0\right)}{\kappa + \left(1 + 2iQ_0 \Delta\omega / \omega_0\right)}$$

We therefore measure a change in the reflection coefficient:

$$\Delta\rho = \rho - \rho_0 \approx \frac{4i\kappa Q_0}{(1+\kappa)^2} \cdot \frac{\Delta\omega}{\omega_0}$$

and including the results of the resonant method this yields:

$$\boxed{E_0(z) = \sqrt{\frac{(1+\kappa)^2}{2\kappa Q_0} \cdot \frac{W}{\alpha_S} \cdot \left|\Delta\rho(z)\right|}}$$

6.4. Determination of the shunt impedance

In chapter two we used the equivalent circuit to introduce the shunt impedance R_S as an oscillating circuit. This leads to the following relation between Voltage U, power dissipation P_V and R_S:

$$R_S = \frac{U^2}{2P_V}$$

(It is worth mentioning that in the consideration of linear accelerators a definition without the factor 2 in the denominator is used. This definition doesn't use the effective value and is also used in the diploma thesis of *Peschke* and the dissertation of *F.O. Müller*!). The accelerating voltage U can be calculated by integrating the electric field along the axis of the resonator:

$$U = \int_0^L E_0(z) \cdot dz$$

To determine the energy gain of a particle we also have to take into account that the field changes during the time, which the particle needs to cross the cavity. In the case of ultra-relativistic particles we have $v \approx c$, thus the time dependence $\cos(\omega t)$ can be expressed as $\cos(\frac{\omega z}{c})$. This effect is often accounted for in the shunt impedance so that we have we have:

$$R_S = \frac{1}{P_V} \cdot \left| \int_{-L/2}^{L/2} E_0(s) \cdot e^{i\frac{\omega_0 s}{c}} \cdot ds \right|^2$$

It can also be described with the delay coefficient:

$$\Lambda = \left| \frac{\int_{-L/2}^{L/2} E_0(s) \cdot e^{i\frac{\omega_0 s}{c}} \cdot ds}{\int_{-L/2}^{L/2} E_0(s) \cdot ds} \right|^2$$

Since the power dissipation is connected to the stored energy via the quality factor, we do not have to know it separately to determine R_S.

We finally have:

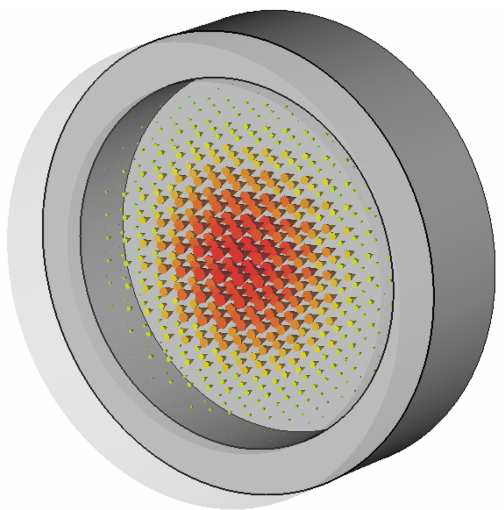
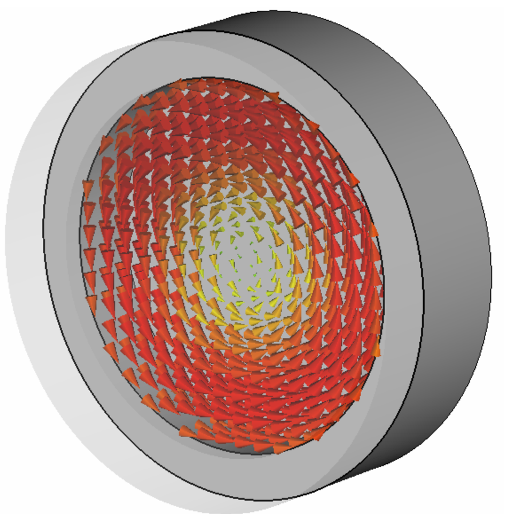
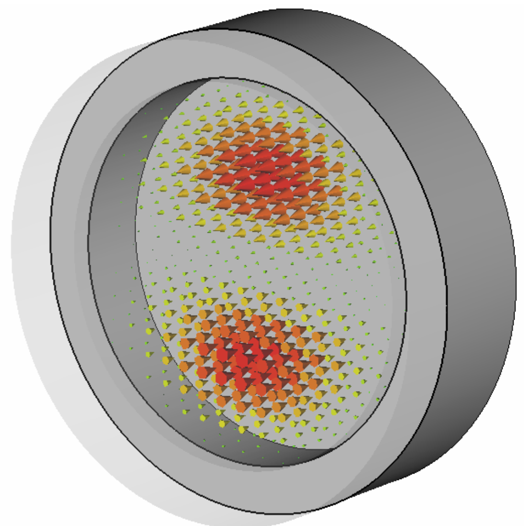
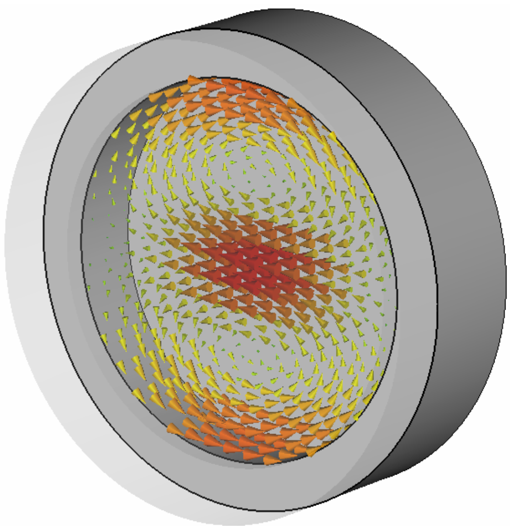
a) resonant method:
$$R_S = \Lambda \cdot \frac{2Q_0}{\omega_0^2 \cdot \alpha_S} \cdot \left| \int_{-L/2}^{L/2} \sqrt{\Delta\omega(z)} \cdot dz \right|^2$$

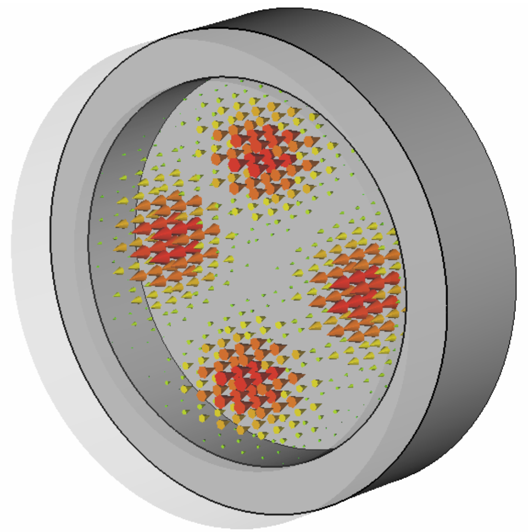
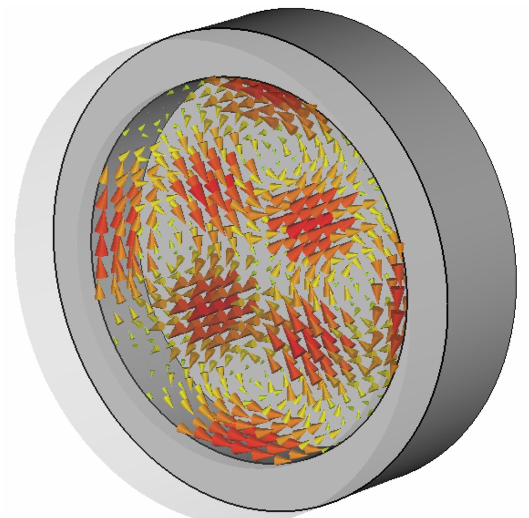
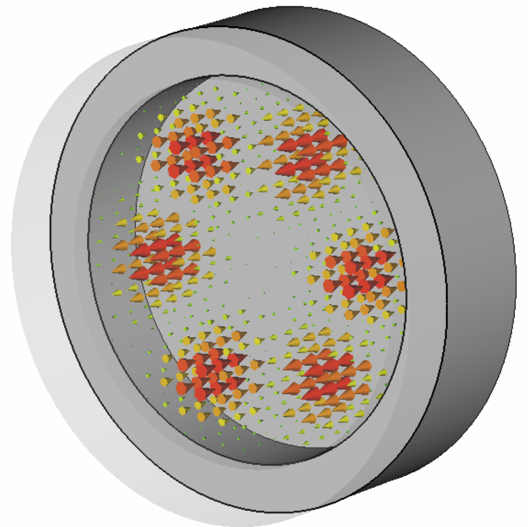
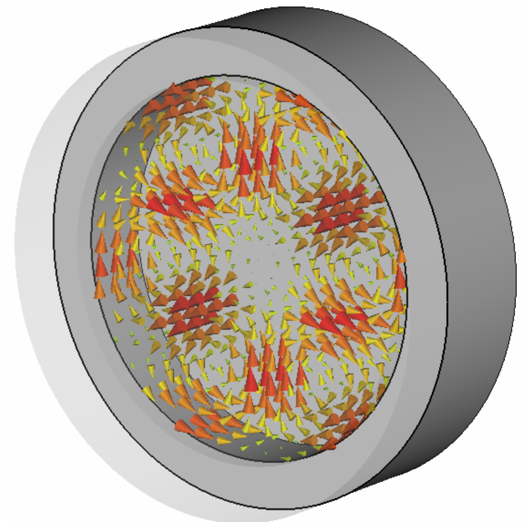
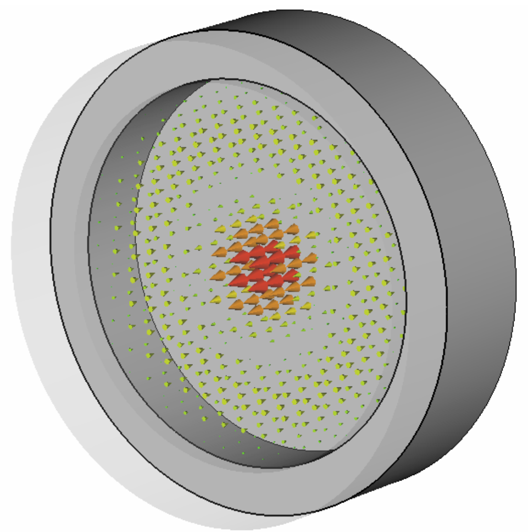
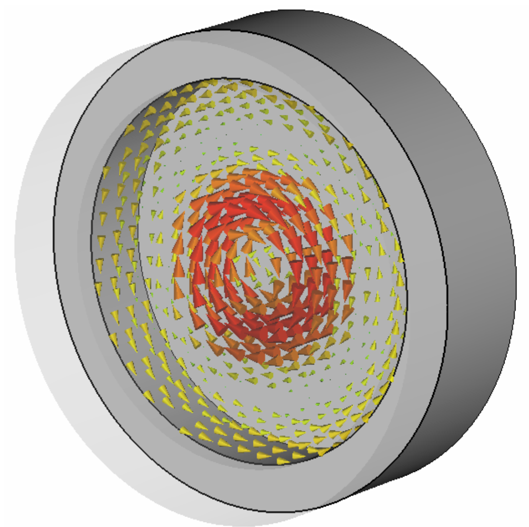
b) non-resonant method:
$$R_S = \Lambda \cdot \frac{(1+\kappa)^2}{2\omega_0 \kappa \alpha_S} \cdot \left| \int_{-L/2}^{L/2} \sqrt{|\Delta\rho(z)|} \cdot dz \right|^2$$

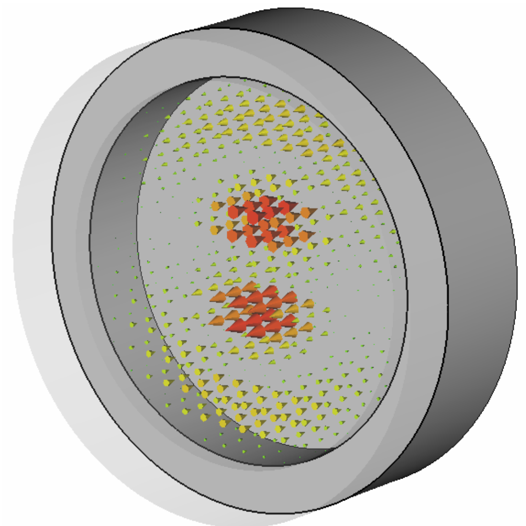
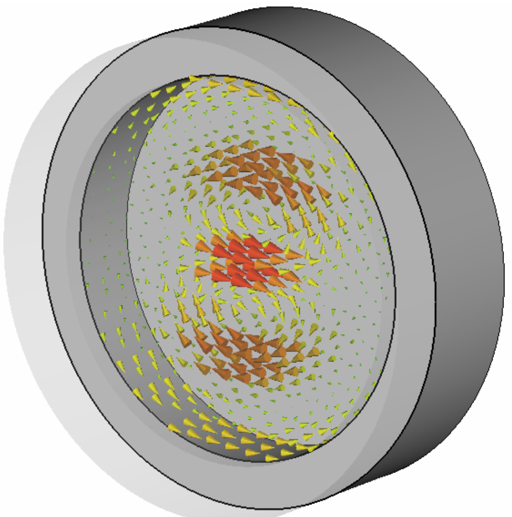
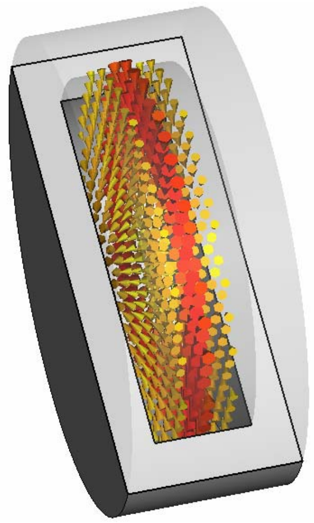
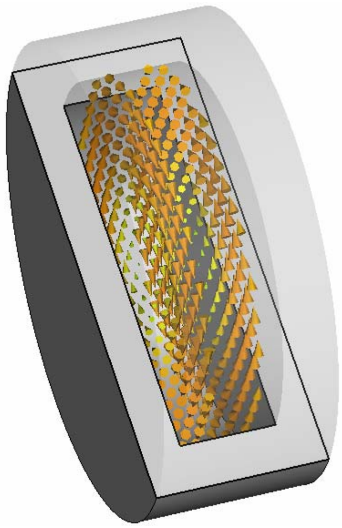
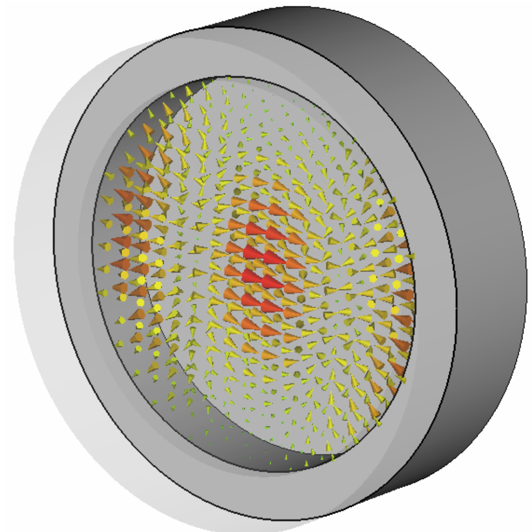
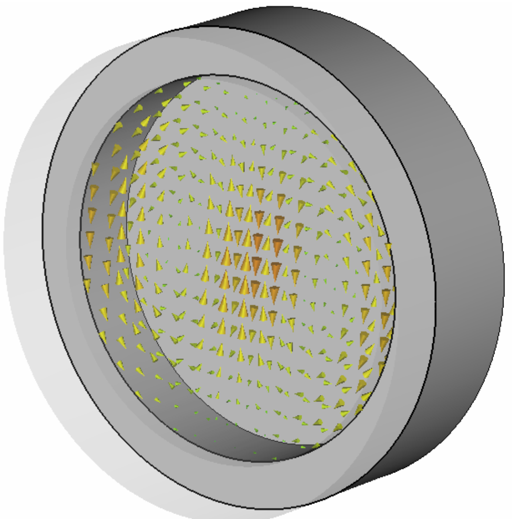
7. Appendix

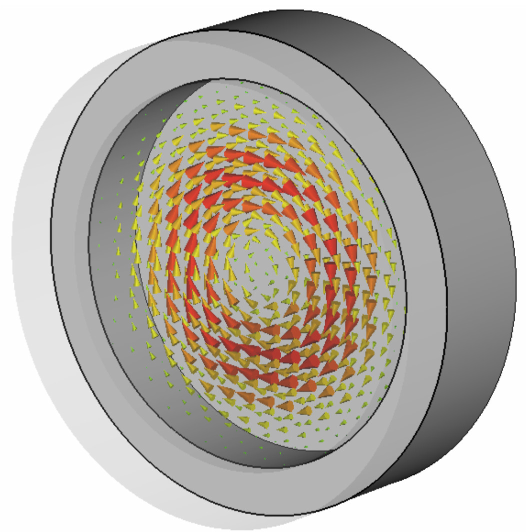
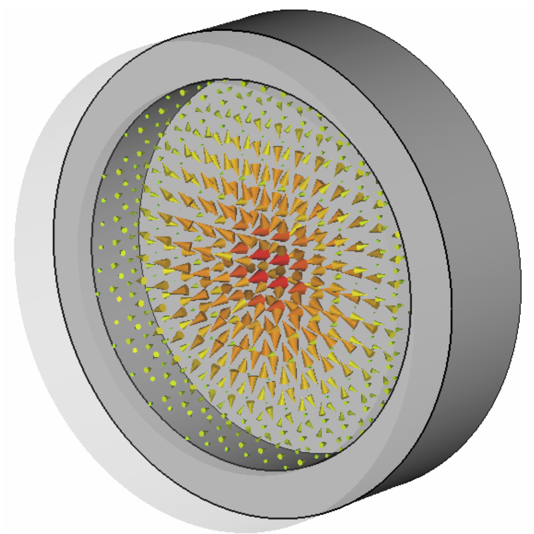
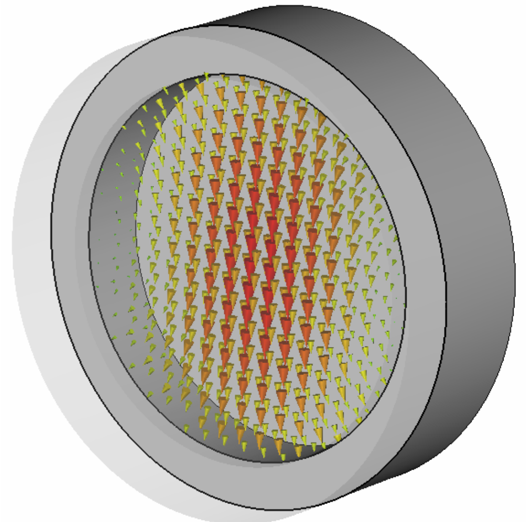
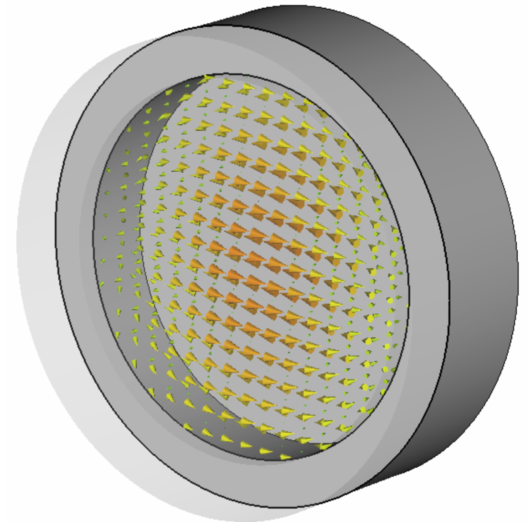
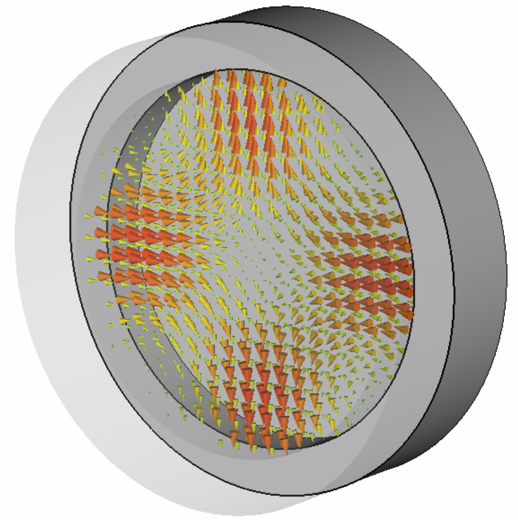
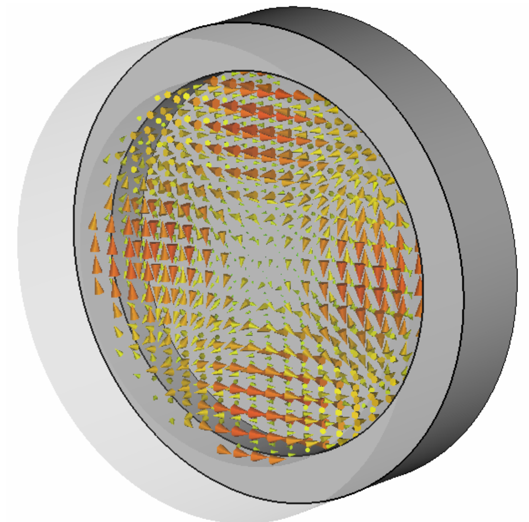
7.1. Field plots of some resonator modes

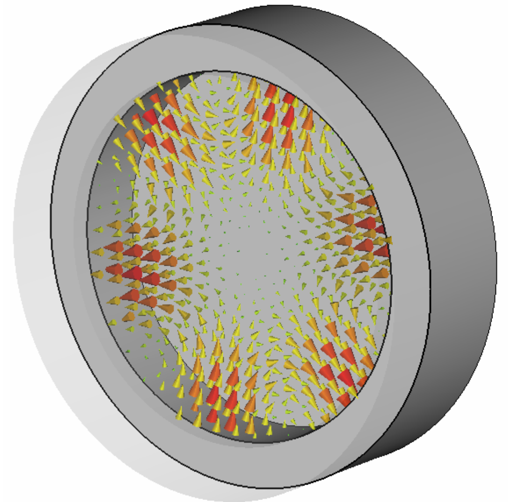
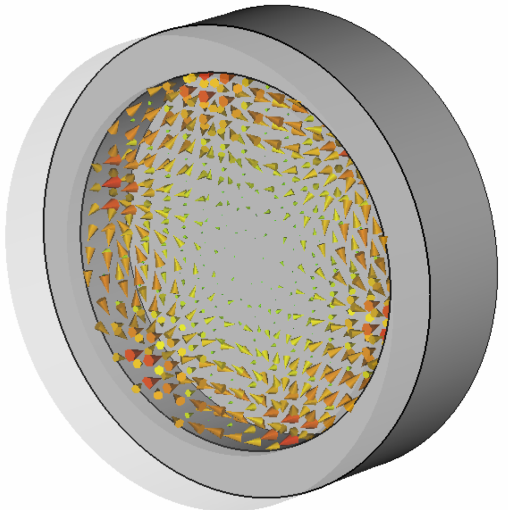
Die untenstehenden Abbildungen zeigen jeweils eine Momentaufnahme der Verteilung der elektrischen und magnetischen Felder für die ersten Resonatormoden (nicht nach Frequenzen geordnet), so wie sie sich aus einer Computersimulation (CST Microwave Studio) ergeben. Gerechnet wurde mit einem zylindrischen, abgeschlossenen Hohlraum, welcher die Abmessungen der im Versuch verwendeten Resonatoren aufweist. Für die Abbildung wurde der Resonator oberhalb einer willkürlich gewählten Schnittebene transparent dargestellt.

mode	electrical field	magnetical field
TM_{010}		
TM_{110}		

mode	**electrical field**	**magnetical field**
TM_{210}		
TM_{310}		
TM_{020}		

mode	**electrical field**	**magnetical field**
TM_{120}		
TM_{011}		
TM_{111}		

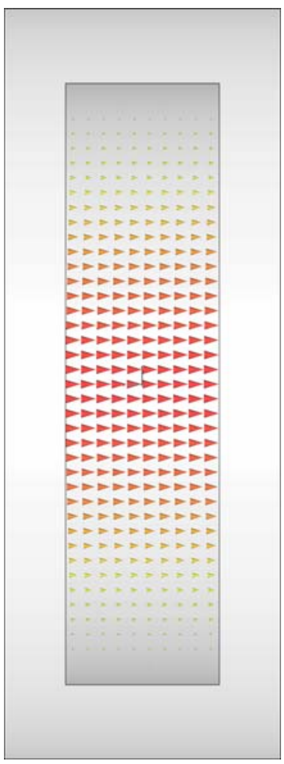
mode	**electrical field**	**magnetical field**
TE_{011}		
TE_{111}		
TE_{211}		

Mode	**Elektrisches Feld**	**Magnetisches Feld**
TE_{311}		

7.2. Influence of lateral holes in the resonator on the field distribution

The resonators used in this experiment have lateral openings which are needed for the bead pull measurements. Comparing to a closed resonator a different field distribution results. The figure shows a comparison of the electrical field of the TM_{010} mode which will have to be determined during the experiment.

resonator without openings



resonator with openings