Laser Beams in Terms of the Courant-Snyder Theory

UBA Workshop 2022

K. Floettmann

6th of June 22

Motivation

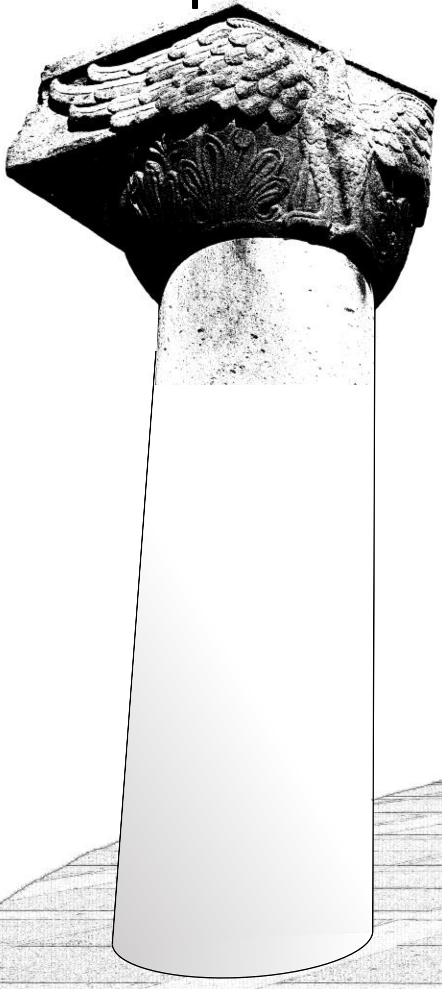
Lasers are integral parts of modern accelerators, e.g.:

- cathode lasers
- for diagnostic purposes
- laser heaters
- in synchronization systems
- to produce THz radiation
- for plasma acceleration
- for direct acceleration/manipulation
- …

We have to deal with them, just as we deal with RF-systems, magnets etc.
However, laser physics is not yet part of the accelerator physics curriculum.

Motivation

Classical Light Optics



imaging problems
- microscopes
- telescopes
- eyeglasses
- …

interference – wave mechanics

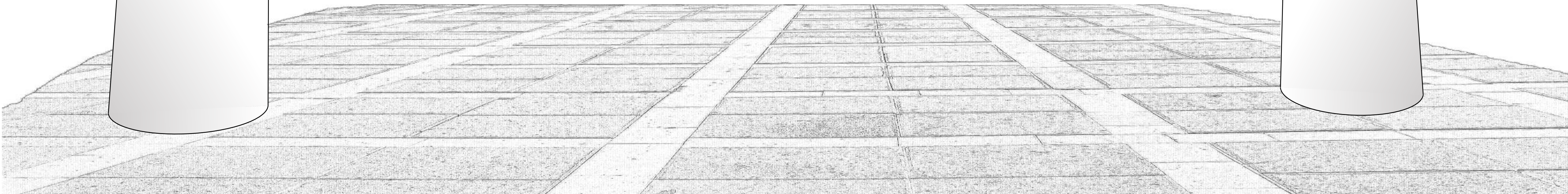
Lasers

Quantum Systems

Electron Microscopes

Optics of Charges Particles

beam physics
- transport and manipulation of beams
- stability (storage rings)
- emittance conservation
- …



Motivation

Classical Light Optics

imaging problems

- microscopes
- telescopes
- eyeglasses
- …

interference – wave mechanics

Lasers

Quantum Systems

Electron Microscopes

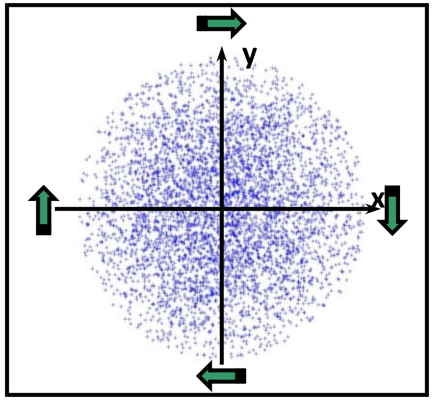
Optics of Charges Particles

BUT:
Laser Beams are B E A M S
and Electron Microscopes
are Accelerators

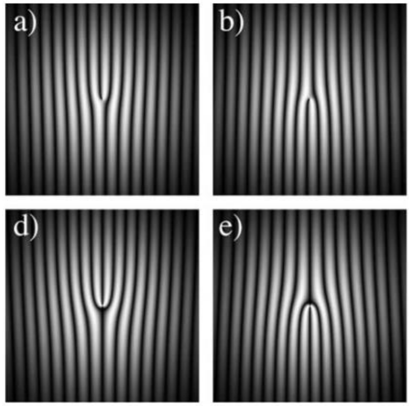
beam physics

- transport and manipulation of beams
- stability (storage rings)
- emittance conservation
- …

Missing communication: Vortex Beams

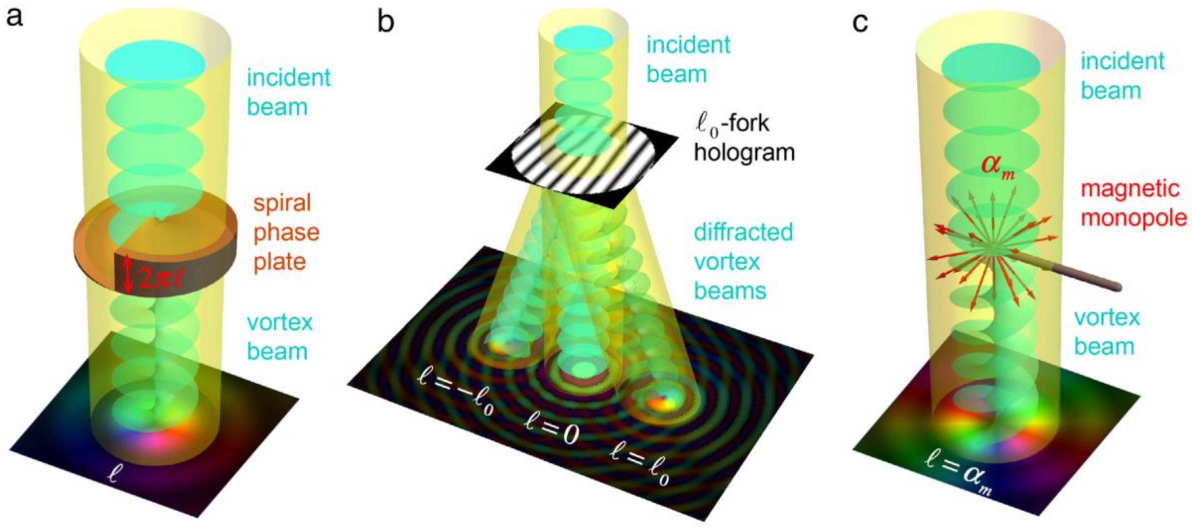


Vortex motion of an electron beam with angular momentum



Fork Holograms for the production of Vortex Beams

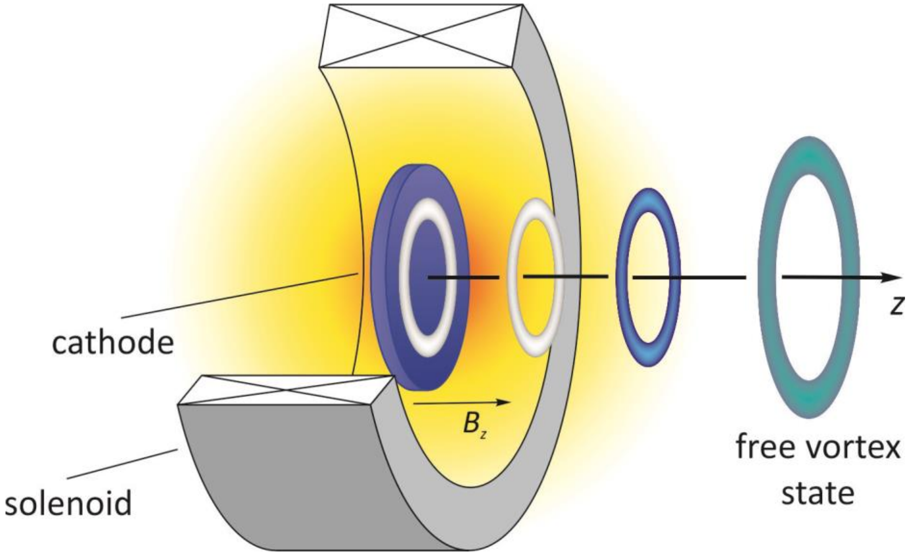
DOI: 10.1119/1.2955792



Basic methods for the generation of electron vortex beams

Theory and applications of free-electron vortex states

DOI: 10.1016/j.physrep.2017.05.006



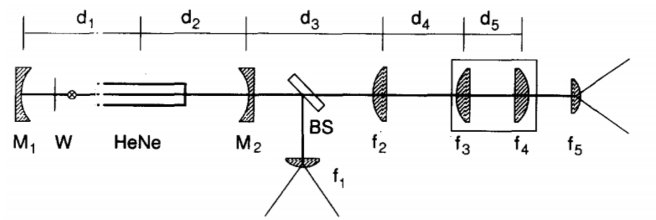
… and the Accelerator Physics method

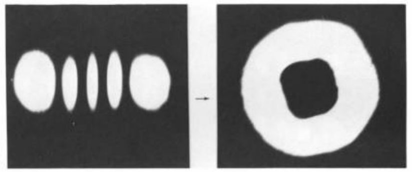
Quantum mechanical formulation of the Busch theorem

DOI: 10.1103/PhysRevA.102.043517

Vortex Beams: Parallel Developments

Laser Physics





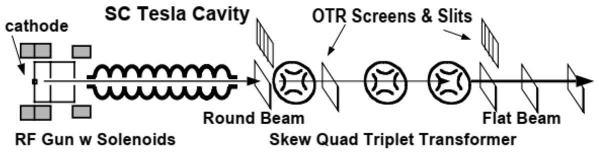
1990 Mode Conversion in Optics

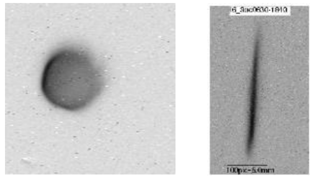
J. Opt. Soc. Am. B **7**, 1034 (1990)
Phys. Rev. A **45**, 8185 (1992)

Pure modes
n<10
Energy ~100 keV
Photons & Electrons

Can we go higher?
Can we go relativistic?
How about other species?
Can we make collision of vortex beams?

Accelerator Physics





1998 Beam Adapter for Accelerators

Y. Derbenev, University of Michigan, Report No. UM-HE-98-04, 1998 (unpublished)
Phys. Rev. ST Accel. Beams **4**, 053501 (2001)

Incoherent beams
n ~10^6
Energy > 20 MeV
Electrons & Ions

Nucl. Instrum. Methods Phys. Res., Sect. A **866**, 36 (2017)

Application in storage rings

accelconf.web.cern.ch/e02/PAPERS/TUPRI044.pd

Adaptation for transverse-longitudinal plane

Phys. Rev. ST Accel. Beams **5**, 084001 (2002)

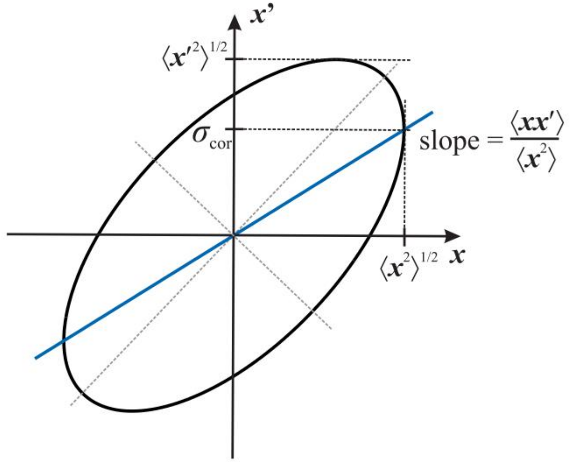
Characteristics of Beams (open boundaries)

- directivity
  - particles or a wave or some form of energy moves predominantly in one direction – *z* coordinate
- finite intensity
  - $A = \iint_{-\infty}^{\infty} \rho(x, y) dxdy < \infty;$ *x*, y perpendicular to *z*, ideally the intensity is a constant of motion
- localization
  - we can define a transverse position and a beam size

$\bar{x} = \int_{-\infty}^{\infty} \rho(x) x dx$ exists at all *z*-positions

$\sigma = \langle x^2 \rangle^{\frac{1}{2}} = \sqrt{\int_{-\infty}^{\infty} \rho(x - \bar{x}) x^2 dx}$ exists at all *z*-positions

Characteristics of Beams – Beam Quality



Every beam can be associated with an rms envelope, independent of all details of the distribution

DOI: 10.1103/PhysRevSTAB.6.034202

Accelerator Physics

$$\varepsilon = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

Laser Physics

$$BPP = \mathbf{M}^2\frac{\lambda}{\pi} = 4\sigma_0\sigma_0' = 4\varepsilon$$

index 0: at focus position

$$\varepsilon = \mathbf{M}^2\frac{\lambda}{4\pi}$$

$\mathbf{M}^2 = 1$: $\sigma_0\sigma_0' = \frac{\lambda}{4\pi}$ Heisenberg's Uncertainty Principle

rms Envelope Equation

$$\sigma = \langle x^2 \rangle^{\frac{1}{2}},$$

$$\sigma_{\text{cor}} = (\sigma)' = \frac{d}{dz}\langle x^2 \rangle^{\frac{1}{2}} = \frac{\langle xx' \rangle}{\langle x^2 \rangle^{\frac{1}{2}}},$$

$$(\sigma)'' = \frac{d^2}{dz^2}\langle x^2 \rangle^{\frac{1}{2}} = \frac{\langle x'^2 \rangle}{\langle x^2 \rangle^{\frac{1}{2}}} + \frac{\langle xx'' \rangle}{\langle x^2 \rangle^{\frac{1}{2}}} - \frac{\langle xx' \rangle^2}{\langle x^2 \rangle^{\frac{3}{2}}}$$

$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} \quad \Longrightarrow \quad \langle x'^2 \rangle = \frac{\varepsilon^2}{\langle x^2 \rangle} + \frac{\langle xx' \rangle^2}{\langle x^2 \rangle}$$

$$(\sigma)'' = \frac{\varepsilon^2}{\langle x^2 \rangle^{\frac{3}{2}}} + \frac{\langle xx'' \rangle}{\langle x^2 \rangle^{\frac{1}{2}}}$$

No assumption about the distribution is required! The rms envelope equation holds for all cases and describes the complete dynamics!

From rms Envelope to Courant-Snyder

first order: single particles and average beam position

second order: rms beam size, rms beam divergence and correlation

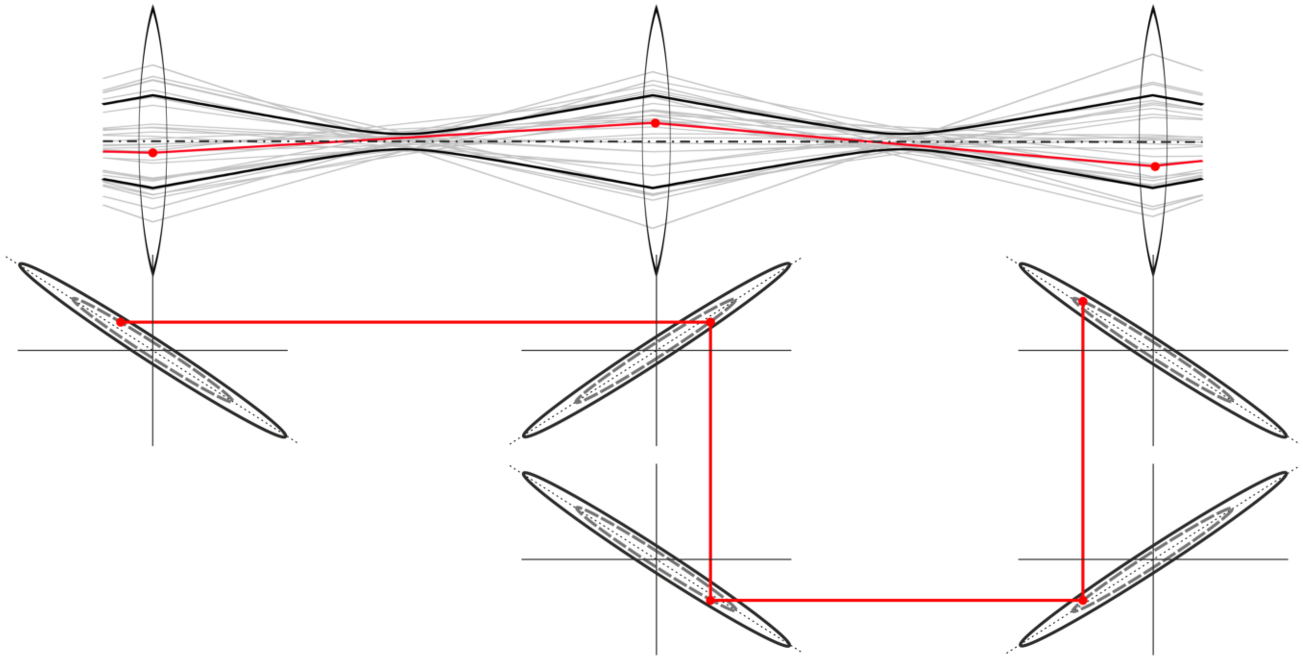
second order: optical functions α, β and γ

first order: phase advance

$\bar{x} \propto \sqrt{\beta} e^{i\phi}$ $\phi = \int \frac{1}{\beta} \mathrm{dz}$

In the Courant-Snyder theory the first order motion (mean beam position) has wave characteristics. This is the key for understanding the stability of storage rings as it allows to discuss resonances!

Courant-Snyder Theory – Phase Advance



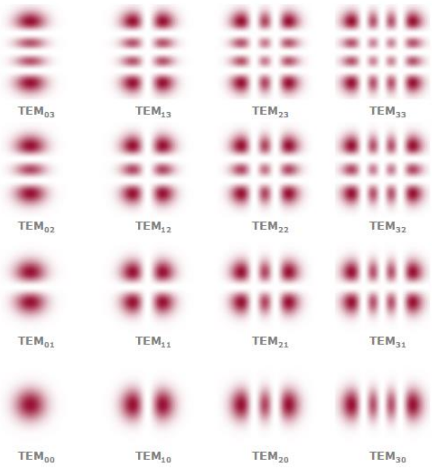
Phase advance in a periodic lens system.
The basic motions in phase space are shearing operations as
$x_{end} = x_{inital} + x'z$ in a drift, and
$x'_{end} = x'_{inital} - Kx$ in a lens

Note: Imaging between two points requires a phase advance of $n \times 180°$!

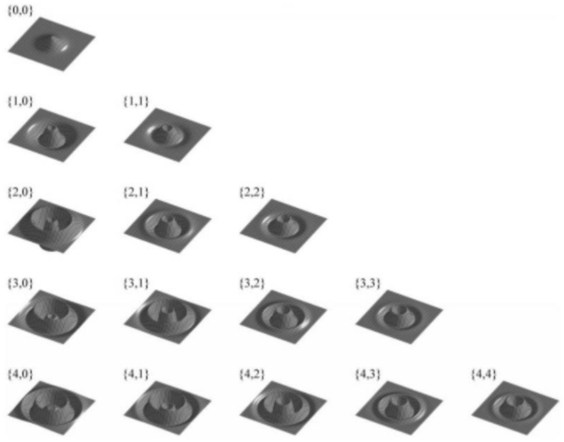
Laser Modes

Modes don't change their shape during propagation.
Modes constitute complete and orthogonal sets of functions describing solutions of the scalar wave equation in paraxial approximation.

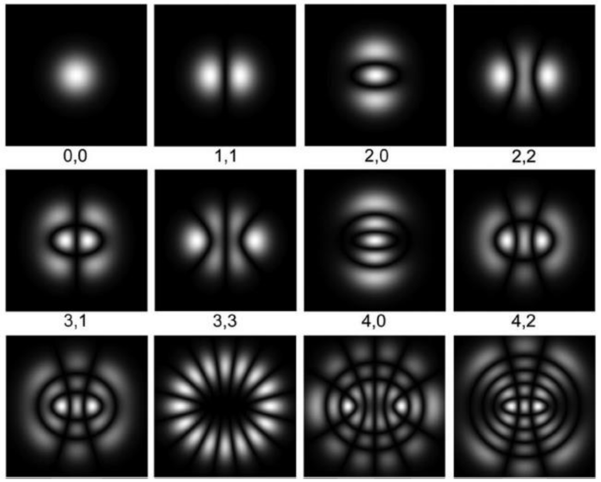
Hermite-Gauss Modes
cartesian coordinates
$\mathbf{M}_x^2 = 2m + 1$
$\mathbf{M}_y^2 = 2n + 1$



Laguerre-Gauss Modes
circular coordinates



Ince-Gauss Modes
elliptical coordinates



Laser Modes: The Amplitude Term

Amplitude × Phase

Hermite-Gauss Modes

$$E(n,m) \propto \frac{1}{w} H_m\left(\sqrt{2}\,\frac{x}{w}\right) H_n\left(\sqrt{2}\,\frac{y}{w}\right) e^{\left(-\frac{x^2+y^2}{w^2}\right)} \times e^{i\theta}$$

w is a beam size parameter, but

$$\sigma_x = \sqrt{2m+1}\,\frac{w}{2}$$

$$\sigma_y = \sqrt{2n+1}\,\frac{w}{2}$$

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = (2x)^2 - 2$$
$$H_3(x) = (2x)^3 - 6(2x)$$
$$H_4(x) = (2x)^4 - 12(2x)^2 + 12$$

The first five Hermite polynomials

$$w = \sqrt{\frac{2\beta}{k}}$$

$$E(n,m) \propto \sqrt{\frac{1}{\beta}} H_m\left(\sqrt{\frac{k}{\beta}}\,x\right) H_n\left(\sqrt{\frac{k}{\beta}}\,y\right) e^{\left(-\frac{k}{\beta}\frac{x^2+y^2}{2}\right)} \times e^{i\theta}$$

$$\sigma_x = \sqrt{\varepsilon_x \beta}$$
$$\sigma_y = \sqrt{\varepsilon_y \beta}$$

Laser Modes: The Phase Term

$\mathbf{M}^2$ term

$$e^{i\theta}:\ \theta = kz - \omega t + \frac{\zeta(x^2+y^2)}{w_0^2(1+\zeta^2)} - (m+n+1)\varphi_G$$

correlated beam divergence

Gouy phase

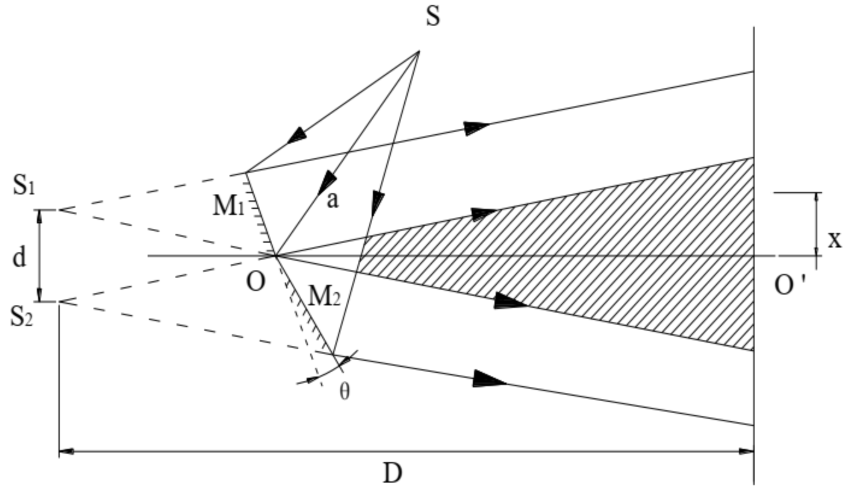
$\zeta = \frac{z}{Z_R}$: Z_R= Rayleigh length

$\sigma^2 = \sigma_0^2\left(1 + \frac{z^2}{Z_R^2}\right)$

$Z_R = \beta_0$= beta function at focus

$$\frac{\zeta(x^2+y^2)}{w_0^2(1+\zeta^2)} = -k\frac{\alpha}{\beta}\frac{x^2+y^2}{2} = -k\left(\frac{\alpha_x x^2}{2\beta_x} + \frac{\alpha_y y^2}{2\beta_y}\right)$$

Laser Modes: The Gouy Phase



Interference experiment with Fresnel Mirror
A. Fresnel 1821

?

L. G. Gouy modified the experiment by inserting a lens though that one of the two beams passes through a focus before the interference screen. Gouy noted, that the phase changed by half a period due to the focus.

L. G. Gouy, 'Sur une propriété nouvelle des ondes lumineuses’, Comp. Rend. Hebdomadaires Séances l’Acad. Sci. **110**, 1251-1253 (1890)

Laser Modes: The Gouy Phase

$$\varphi_G = \arctan(\zeta) = \int \frac{1}{1+\zeta^2} d\zeta = \int \frac{1}{\beta} dz = \phi$$

$$\zeta = \frac{z}{\beta_0}; \quad d\zeta = \frac{dz}{\beta_0}$$

The Gouy phase is equivalent to the Courant-Snyder phase advance .

The Gouy phase not an anomaly, but describes the normal development of the phase in a beam (in contrast to a plane wave which is not a beam).

Another beautiful connection of wave and ray optics.

Equivalence of Gouy and Courant-Snyder phase

PHYSICAL REVIEW A **102**, 033507 (2020)
DOI: 10.1103/PhysRevA.102.033507

Summary

$$E \propto \frac{1}{w} H_m\left(\sqrt{2}\frac{x}{w}\right) H_n\left(\sqrt{2}\frac{y}{w}\right) \operatorname{Exp}\left[-\frac{x^2+y^2}{w^2}\right] \times \operatorname{Exp}\left[i\left(kz-\omega t+\frac{\zeta(x^2+y^2)}{w_0^2(1+\zeta^2)}-(m+n+1)\varphi_G\right)\right]$$

- meaning of w unclear
- round beams, i.e. $w_x = w_y$
- no astigmatism
- optical systems are difficult, because ζ and w_0 refer to a focus
- transfer of the Gouy phase in optical systems is often ignored

$$E \propto F_x F_y$$

$$F_x = \frac{1}{\sqrt[4]{\beta_x}} H_m\left(\sqrt{\frac{k}{\beta_x}}x\right) \operatorname{Exp}\left[-\frac{k\,x^2}{2\beta_x}\right] \times \operatorname{Exp}\left[i\frac{1}{2}\left(kz-\omega t-k\left(\frac{\alpha_x x^2}{\beta_x}\right)-\mathbf{M}_x^2\phi_x\right)\right]$$

F_y equivalent

- elliptical beams, i.e. $\beta_x \neq \beta_y$
- astigmatism, i.e. $\alpha_x \neq \alpha_y$
- decoupled phase advance, i.e. $\phi_x \neq \phi_y$
- α, β and ϕ are well-defined in optical systems


Thank you for your attention !