A NEW APPROACH TO SOLVING THE PROBLEM OF AN EXTENDED HELICAL UNDULATOR

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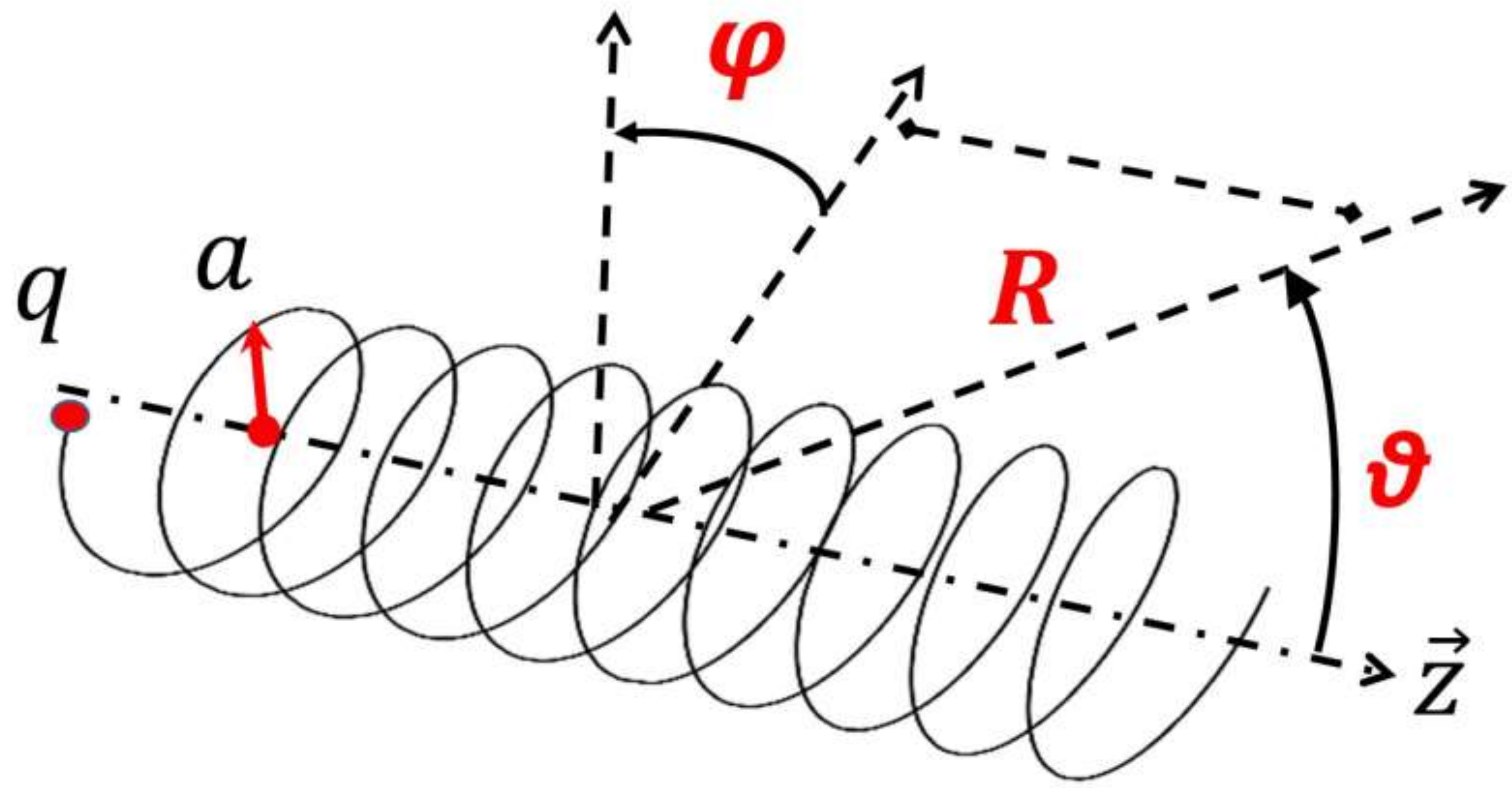
Content

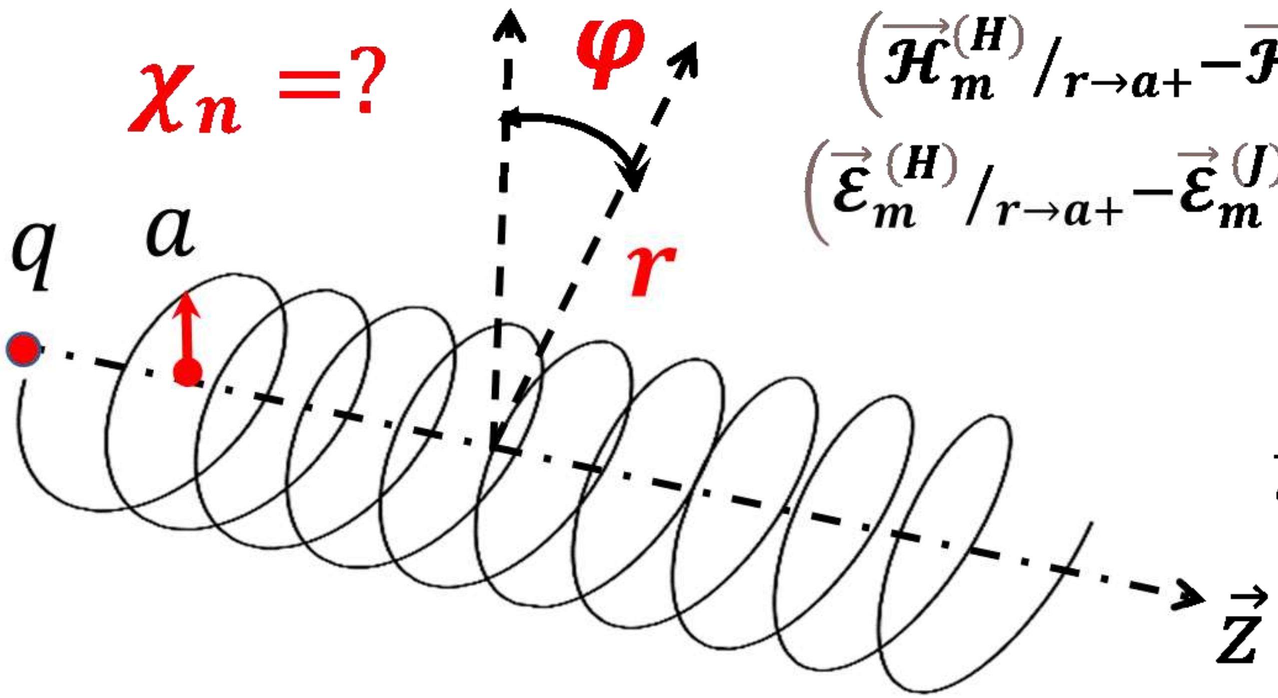
Abstract

An exact solution for the radiation field of a particle in a helical undulator, valid for an arbitrary point in space and an arbitrary particle energy, was obtained by the partial domain method, generalized for the case of a spiral motion of the particle. The interface between the regions is a cylindrical surface containing the spiral trajectory of the particle. A comparison is made with the existing solution, which is valid in the far zone at high particle energies.

Statement of problem



Spherical frame, commonly used



$$\left(\overrightarrow{\mathcal{H}}_m^{(H)}/_{r\to a+} - \overrightarrow{\mathcal{H}}_m^{(J)}/_{r\to a-}\right) \times \vec{e}_r = \chi_n \vec{J}$$

$$\left(\vec{\mathcal{E}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{E}}_m^{(J)}/_{r\to a-}\right) \cdot \vec{e}_r = -\chi_n q/\varepsilon_0$$

$$\vec{\mathcal{E}} = \begin{cases} \sum_{m=-\infty}^{\infty} \vec{\mathcal{E}}_m^{(J)}, r < a \\ \sum_{m=-\infty}^{\infty} \vec{\mathcal{E}}_m^{(H)}, r > a \end{cases}$$

Cylindrical frame, used in this work

FIELD PRESENTATION

$$\vec{\mathcal{E}}_m^{(Z)} = \mathcal{A}_m^{(Z)} \vec{E}_{m,TM}^{(0,,Z)} + \mathcal{B}_m^{(Z)} \vec{E}_{m,TE}^{(0,Z)}$$

$$Z_0 \overrightarrow{\mathcal{H}}_m^{(Z)} = \mathcal{A}_m^{(Z)} \vec{H}_{m,TM}^{(0,Z)} + \mathcal{B}_m^{(Z)} \vec{H}_{m,TE}^{(0,Z)}$$

$$\vec{\mathcal{E}} = \begin{cases} \sum_{m=-\infty}^{\infty} \vec{\mathcal{E}}_m^{(J)} \\ \sum_{m=-\infty}^{\infty} \vec{\mathcal{E}}_m^{(H)} \end{cases}, \quad \overrightarrow{\mathcal{H}} = \begin{cases} \sum_{m=-\infty}^{\infty} \overrightarrow{\mathcal{H}}_m^{(J)} & r < a \\ \sum_{m=-\infty}^{\infty} \overrightarrow{\mathcal{H}}_m^{(H)} & r > a \end{cases}$$

$$\vec{E}^{(0,Z)}_{m,TM} = -\nu^{-2}\,\mathrm{rot}\,\vec{R}, \qquad Z_0\vec{H}^{(0,Z)}_{m,TM} = jk\nu^{-2}\vec{R},$$

$$Z_0\vec{H}^{(0,Z)}_{m,TE} = -\nu^{-2}\mathrm{rot}\,\vec{R}, \qquad \vec{E}^{(0,Z)}_{m,TE} = -jk\nu^{-2}\vec{R}$$

$$\vec{R} = \vec{e}_z \times \vec{\nabla}P^Z, \qquad P^Z = Z_m e^{j(m\varphi+pz-\omega t)}, \qquad k = \omega/c$$

If $\boldsymbol{Z = J,\ Z_m = J_m(\nu_m r)}$ and if $\boldsymbol{Z = H,\ Z_m = H_m^{(1)}(\nu_m r)}$, where J_m and $H_m^{(1)}$ are the Bessel function and the Hankel function of the first kind; $Z_0 = (\varepsilon_0 c)^{-1}$ is the impedance of free space and ε_0 is the dielectric constant of vacuum; $\boldsymbol{p_m}$ and $\boldsymbol{\nu_m = \sqrt{\omega^2/c^2 - p_m^2}}$ are the longitudinal and transverse eigenvalues of the m^{th} mode. In the case of linear motion of the particle $p_m = \omega/v$ (v is the total velocity of the particle) and $\nu_m = j\omega/v\gamma$ (j is the imaginary unit), while for the helical motion $p_m = (\omega - m\omega_0)/V$ (V is longitudinal component of the particle velocity) and

$$\boldsymbol{\nu_m = \sqrt{\omega^2/c^2 - (\omega - m\omega_0)^2/V^2}}$$

Boundary conditions on the cylindrical surface, containing particle trajectory

J. A. Stratton, "*Electromagnetic Theory*", New York, NY, USA, McGRAW HILL, 1941

$$\left(\vec{\mathcal{E}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{E}}_m^{(J)}/_{r\to a-}\right) \times \vec{e}_r = 0$$

$$\left(\vec{\mathcal{H}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{H}}_m^{(J)}/_{r\to a-}\right) \times \vec{e}_r = \chi_n \vec{J}$$

$$\left(\vec{\mathcal{E}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{E}}_m^{(J)}/_{r\to a-}\right) \cdot \vec{e}_r = -\chi_n q/\varepsilon_0$$

$$\left(\vec{\mathcal{H}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{H}}_m^{(J)}/_{r\to a-}\right) \cdot \vec{e}_r = 0$$

Solutions for amplitudes

$$\begin{matrix}\mathcal{A}_m^{(I)} \\ \mathcal{A}_m^{(H)}\end{matrix} = q\frac{\pi a}{2\varepsilon_0 V\omega}\chi_m f_m \begin{cases} H_m^{(1)}(a\nu_m) \\ J_m(a\nu_m)\end{cases} \qquad \begin{matrix}\mathcal{B}_m^{(I)} \\ \mathcal{B}_m^{(H)}\end{matrix} = jq\frac{\pi a^2\omega_0}{2\varepsilon_0 c}\chi_m \nu_m \begin{cases} H_m^{(1)\prime}(a\nu_m) \\ J_m'(a\nu_m)\end{cases}$$

$$f_m = \omega\left(\omega_0 m - \omega/\gamma_z^2\right), \qquad \gamma_z^2 = \left(1 - V^2/c^2\right)^{-1}$$

COUPLING OF TM AND TE MODES

Examining the components of the obtained expressions (3, 6) for the fields for the TM and TE modes separately, we find that the transverse components have a divergence of the order of ν_m^{-2} at $\nu_m \to 0$ (without taking into account the features of the yet unknown function χ_m) for arbitrary integer values $m > 0$, but in their superposition (2) these singularities for $m > 1$ are mutually compensated. For example, the radial magnetic TM and TE components in the vicinity of $\nu_m = 0$ have singularities equal in magnitude and opposite in sign:

$$\left.\begin{matrix} \mathcal{A}_m^{(I)} H_{m,TM_r}^{(0,,I)} \\ \mathcal{A}_m^{(H)} H_{m,TM_r}^{(0,,H)} \end{matrix}\right\}_{\nu_m \to 0} = \left.\begin{matrix} -\mathcal{B}_m^{(I)} H_{m,TE_r}^{(0,,I)} \\ -\mathcal{B}_m^{(I)} H_{m,TE_r}^{(0,,I)} \end{matrix}\right\}_{\nu_m \to 0} = jm\chi_m \frac{q}{2} p_m \omega_0 \left(\frac{r}{a}\right)^{\pm m-1} \nu_m^{-2}$$

Only when $m = 1$ is the logarithmic divergence preserved (at $\nu_1 = 0$). So, for the radial electrical components:

$$\left.\begin{matrix} \mathcal{A}_1^{(I)} E_{1,TM_r}^{(0,,I)} \\ \mathcal{A}_1^{(H)} E_{1,TM_r}^{(0,,H)} \end{matrix}\right\}_{\nu_1 \to 0} = \mp \chi_m \frac{q}{2\varepsilon_0} p_1^2 \frac{\omega_0}{\omega} \begin{Bmatrix} 1 \\ a^2/r^2 \end{Bmatrix} \nu_1^{-2} + \chi_m \frac{q}{4\varepsilon_0} a^2 p_1^2 \frac{\omega_0}{\omega} \begin{Bmatrix} ln(a\nu_1/2) \\ ln(r\nu_1/2) \end{Bmatrix}$$

$$\left.\begin{matrix} \mathcal{B}_1^{(I)} E_{1,TE_r}^{(0,,I)} \\ \mathcal{B}_1^{(H)} E_{1,TE_r}^{(0,,H)} \end{matrix}\right\}_{\nu_1 \to 0} = \pm \chi_m \frac{q\omega\omega_0}{2c^2\varepsilon_0} \begin{Bmatrix} 1 \\ a^2/r^2 \end{Bmatrix} \nu_1^{-2} + \chi_m a^2 \frac{q\omega\omega_0}{4c^2\varepsilon_0} \begin{Bmatrix} ln(a\nu_1/2) \\ ln(r\nu_1/2) \end{Bmatrix}$$

This divergence should be eliminated by an appropriate selection of the factor χ_m. Note that the longitudinal components do not have any singularities. Thus, TM and TE modes are mutually coupled and cannot be generated separately.

DETERMINATION OF χ_m

RADIATION POWER SPECTRAL DENSITY DISTRIBUTION

Comparison

New expression

$$\boxed{\frac{dJ(\omega)}{d\omega}=\frac{q^2N}{2\varepsilon_0 c}\sum_{m=1}^{\infty}Q_m X_m \tilde{u}\left(S_m^2\right)}$$

$$X_m=\tilde{\omega}\left\{\frac{\tilde{D}_m^2}{S_m}J_m^2(\tilde{y}_m)+\beta_\varphi^2 J_m'^2(\tilde{y}_m)\right\}$$

$$Q_m=4\pi^3\chi_m^2\,a^2\omega/c\nu_m$$

$$\tilde{D}_m=m-\tilde{\omega}/\gamma_z^2,\qquad \tilde{B}_m=m(m-\tilde{\omega})$$

$$S_m=\tilde{\omega}\tilde{D}_m-\tilde{B}_m,\quad \tilde{y}_m=\frac{\beta_\varphi}{\beta_z}\sqrt{S_m},$$

$$\tilde{\omega}=\omega/\omega_0,$$

Taken from:

B. M. Kincaid, Journal of Appl. Phys., vol. 48, p. 2684, 1977.

$$\boxed{\frac{dI(\omega)}{d\omega}=\frac{Nq^2K^2\tilde{r}}{\varepsilon_0 c}\sum_{m=1}^{\infty}Y_m\tilde{u}\left(\tilde{\alpha}_m^2\right)}$$

$$Y_m=J_m'^2(\tilde{x}_m)+(\tilde{\alpha}_m/K-m/\tilde{x}_m)^2J_m^2(\tilde{x}_m)$$

$$\tilde{\alpha}_m^2=m/\tilde{r}-1-K^2,\tilde{x}_m=2K\tilde{r}\tilde{\alpha}_m,$$

$$\tilde{r}=\omega/2\gamma^2\omega_0,$$

$\tilde{u}(x)$ is a unit step function

A coincidence occurs at

$\gamma \gg 1$ and $2\omega \gg m\omega_0$ with $Q_m = 1$

Thus

$$\chi_m = \frac{1}{2a\pi^{3/2}}\left(\frac{cv_m}{\omega}\right)^{1/2}$$

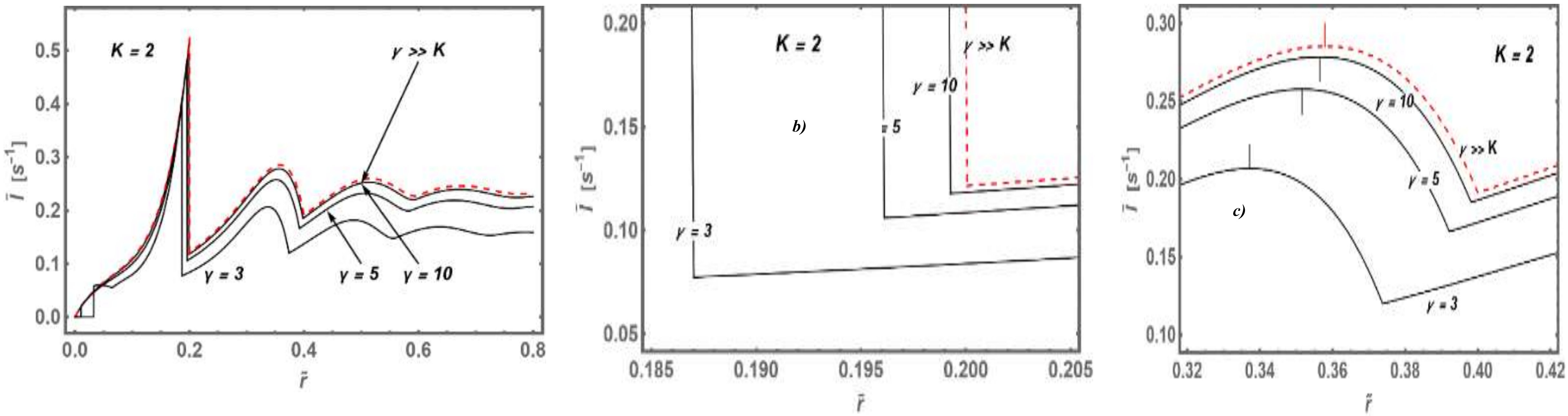
Kincaid's formula has one remarkable property: it allows the spectra to depend on the generalized parameter $\tilde{r} = \omega/2\gamma^2\omega_0$, leaving only the parameter K free. This is achieved in the approximation of very high energies and far from frequencies that are multiples of the rotation frequency ($2\omega \gg m\omega_0$).

New Formula, after substitution of the approximate expression for the longitudinal velocity

$$V = c\sqrt{(1-\gamma^{-2})(1-K^2\gamma^{-2})},$$

is a refined version of Kincaid's formula: in this case, the requirement for γ is relaxed and the constraint $2\omega \gg m\omega_0$ is completely eliminated. This option with the artificial introduction of a functional dependence on $\tilde{r}$, as in formula Kincaid's formula , retains the parametric dependence on K, while acquiring an additional parametric dependence on γ.

NUMERICAL COMPARISON



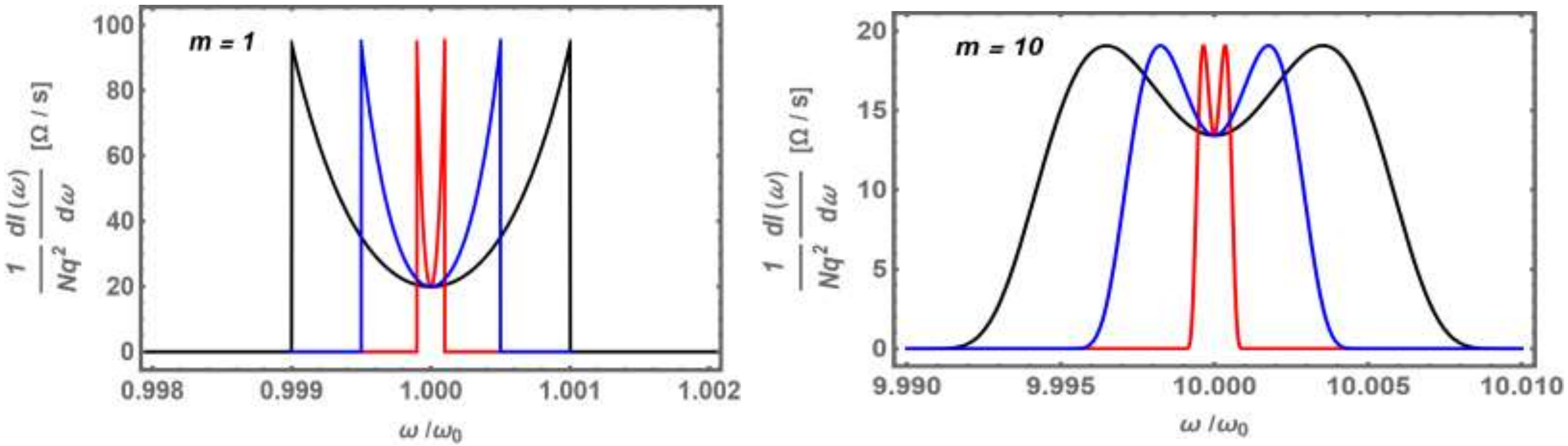
Spectral distributions of the radiation energy density of a particle moving along a spiral trajectory. The red dotted curve is calculated using Kincaid's formula; the black solid curves, calculated for three different values of γ, corresponds to New formula; a) full spectrum; b), c) selected parts of the spectrum; $K = 2$; $\tilde{I}$ – distributions normalized to $Nq^2/\varepsilon_0 c$; $\tilde{r} = \omega/2\gamma^2\omega_0$.

Transition to quasi-discrete spectrum

The allowed frequency regions of the spectrum of each mode are determined from the condition $S_m > 0$,, which implies their narrowing as the longitudinal propagation speed decreases:

$$\frac{m}{1+\beta_z} < \tilde{\omega} < \frac{m}{1-\beta_z}$$

In this case, the spectrum passes from continuous to fragmentary.



Spectral density of radiation energy distribution of a particle moving along a spiral trajectory at low longitudinal velocities: $\beta_z = 0.0001\beta_\gamma$ (red), $0.0005\beta_\gamma$ (blue), $0.001\beta_\gamma$ (black); First (left) and tenth (right) harmonics. Particle energy E=30MeV (γ≈58.71).

CONCLUSION

The use of a cylindrical coordinate system directly related to the particle trajectory made it possible to use the partial domain method for a uniform description of radiation fields both near and far from the particle trajectory and to take into account the discontinuity of the field components on the cylindrical surface containing the particle trajectory, generalizing it to the case of a helical motion of the particle.

The main results are: 1) For the radiation of a particle moving along a spiral trajectory with a constant longitudinal velocity and a fixed rotation frequency, expressions for the fields are obtained that are valid at any point in space.

2) The mutual compensation of singularities, present in TM and TE modes with the same order, has been proven. 3) A refined formula for the distribution of the spectral energy density of the radiation of a helical undulator was obtained. 4) The principal possibility of sampling the emission spectrum of a helical undulator has been demonstrated

THANK YOU！